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THE OBLIQUE REFLEXION OF LONG WIRELESS WAVES FROM THE IONOSPHERE AT PLACES WHERE THE EARTH'S MAGNETIC FIELD IS REGARDED AS VERTICAL

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The calculation of reflexion coefficients for long wireless waves incident obliquely on the ionosphere requires an exact solution of the differential equations governing the propagation of electromagnetic waves in the ionosphere. Equations are developed for the electromagnetic field in a horizontally stratified medium of varying electron density, the presence of a vertical external magnetic field and also the collision frequency of the electrons with neutral molecules being taken into account. Provided certain inequalities hold amongst these ionospheric characteristics, the ionosphere splits up effectively into two regions, in each of which the differential equations of wave propagation approximate to simpler forms. If a model ionosphere is chosen in which the ionization density increases exponentially with height, and the collision frequency is assumed constant over the range of height responsible for reflexion, the equations for the two regions can be solved exactly. The solution for the lower region is expressed in terms of hypergeometric functions, and that for the upper region in terms of generalized confluent hypergeometric functions. Exact expressions in terms of factorial functions can then be deduced for the reflexion coefficients of both regions separately. Moreover, these coefficients can be combined, with due allowance for the path difference between the two regions, to give the overall reflexion coefficients for the effect of the ionosphere as a whole on an incident wave. A suitable definition is given for the apparent height of reflexion in terms of the phase of the reflected wave. The results of the theory are illustrated in graphical form for a particular model ionosphere approximating to the 'tail' of a Chapman region, and a brief comparison with experimental observations concludes the paper.

CONTENTS

1. Introduction

In any theory of the propagation of long radio waves in the ionosphere, difficulties arise when the wave-length is so great that the properties of the ionosphere change appreciably in distances comparable with 1 wave-length. Under these conditions it is no longer possible to solve the propagation equations by using the 'slowly varying' approximation developed by Appleton (1937) and Booker (1934, 1935); and the full formal wave solution, in the presence of the earth's steady magnetic field, presents considerable difficulties.

In two papers, Wilkes has gone some way towards solving the problem. In his first paper (Wilkes 1940) he considered only the case of waves incident vertically on the ionosphere, and in the theory he took the earth's magnetic field as vertical. He showed how to calculate the characteristics of the reflected wave for the two cases where (*a*) the density of the electrons increased as the square of the height, and the frequency with which they collided with neutral molecules was independent of the height, and (*b*) the electron density was constant but the collision frequency decreased inversely as the height.

In his second paper (Wilkes 1947) he outlined an extension of the theory to include the case of oblique incidence, again taking the earth's magnetic field as vertical. He showed generally that, with certain inequalities holding throughout the ionosphere, it was convenient theoretically to consider two separate regions; in the lower of these, which he named the 'transitional region', the collision frequency of the electrons was of importance; in the upper region, which he termed the 'reflecting region', the collision frequency could be assumed to be negligible. Wilkes did not attempt to predict theoretically the effect of the transitional region on a wave passing through it but suggested that absorption effects rather than reflexion would take place. Since the analysis of this paper will show that reflexion can occur from this transitional region, it has been considered preferable in this paper to rename these two regions, by calling the lower transitional region 'region I' and the upper reflecting region 'region II'. For vertical incidence, region I disappears because of the dependence of the equations of propagation on the sine of the angle of incidence. It also disappears for horizontally polarized waves.

To illustrate the general theory Wilkes took a model in which the electron density increased linearly with height. He dealt in detail only with the reflecting region, and he was able to deduce, in general terms, a solution for the propagation of the wave in it. The solution was not expressible in terms of tabulated functions, but as a set of integrals whose asymptotic expansions were given; the reflexion coefficients could not be evaluated without lengthy numerical computation. Though he gave no graphical representation of his results, he was able to show that no radical change would be expected as the angle of incidence was altered from vertical to oblique.

In this paper, Wilkes's method of dividing the ionosphere into two distinct regions is used, and the external magnetic field is again regarded as vertical. A model is assumed in which the ionization density increases upwards exponentially with height and the collision frequency is effectively constant over the rather narrow region where it plays an important role. It is shown that the equations governing the oblique propagation of radio waves in both regions can be solved analytically in such a manner that the reflexion coefficients can be expressed exactly in terms of factorial functions which offer no computational difficulty. This solution is made possible by a special change of independent variable (see § 4, equation (8), and § 7, equation (4)) which yields equations having hypergeometric functions as solutions for region I and generalized confluent hypergeometric functions as solutions for region II.

The nature of these two regions is discussed in a preliminary way in § 3. In §§ 4 to 9 expressions for the reflexion and transmission coefficients of the two regions separately are derived, when the waves are incident obliquely. § 10 deals with waves incident from below both regions, these waves being partly reflected and partly transmitted by the lower region I and reflected by the upper region II. Expressions for the overall reflexion coefficients are given. From the phase of the wave finally returned from the two regions together, an expression is deduced for the 'apparent height of reflexion'.

In § 13 the results of calculations are given in the form of a series of graphs drawn for a particular model ionosphere for the frequencies of 16 and 80 kc/s. The ionosphere is chosen to approximate closely to the 'tail' of a Chapman region such as might constitute the lowest part of the E layer; the numerical parameters assumed for its description are discussed in § 12. Also in § 13 a comparison is made in outline with the results obtained by experiment.

It has not been found possible to explain the way in which the absorption of the reflected wave varies with frequency, and it must still be assumed, as suggested by Wilkes (1940), that there is an important absorbing region at a level below those considered in this paper. If this absorption occurs at heights where the collision frequency is great enough ($Z \gg |Y|$),* it will not affect the polarization appreciably. Stanley (1950) has recently outlined an extension to Wilkes's theory, for the case of vertical incidence, and has shown that by including the effect of an extended lower region of weak ionization it is possible to explain the way in which absorption varies with frequency. Using Wilkes's treatment of the oblique-incidence equations, it is not possible to repeat Stanley's analysis for an ionosphere where the electron density varies exponentially with height, since the values he needs for the collision frequency violate the inequalities which must hold for Wilkes's method.

2. Notation

We shall use the following notation:

x, y, z Cartesian co-ordinates, z measured vertically upwards from the ground.
E $= (E_x, E_y, E_z)$ electric field components of the wave.
H $= (H_x, H_y, H_z)$ magnetic field components of the wave.
c = velocity of electromagnetic waves *in vacuo*.
m = mass of electron.
e = positive numerical value of charge on electron.
N = electron density.
H_e = earth's magnetic field (assumed vertical).
p $= 2\pi \times$ frequency of waves.
ν = electronic collision frequency.
ϵ_0 = dielectric constant of free space.
μ_0 = permeability of free space.
X $= Ne^2/m\epsilon_0 p^2 = p_0^2/p^2$.
Y $= eH_e\mu_0/mp = p_H/p$.
Z $= \nu/p$.
k $= 2\pi/\lambda = p/c$.

* The nomenclature is explained in § 2.

θ = angle of incidence of electromagnetic waves (in x, z plane, measured from z-axis).
l $= \sin^2\theta/[\cos^2\theta - X/(1-\mathrm{i}Z)]$.
H = scale height of atmosphere.
z_0 = actual height of maximum ionization.
z_1 = height where $|X/(1-\mathrm{i}Z)| = 1$.
z_2 $= z_3 + \frac{2}{\gamma}\log_e(2\gamma/k)$.
z_3 = height where $|X/Y| = 1$.
α $= [\partial(\log X)/\partial z]_{z=z_1}$, assumed effectively constant throughout region I.
γ $= [\partial(\log X)/\partial z]_{z=z_3}$, assumed effectively constant throughout region II.
ϵ $= \arg[X/(1-\mathrm{i}Z)]_{z=z_1}$, assumed effectively constant throughout region I.
v $= \exp(\mathrm{i}\epsilon)\exp[\alpha(z-z_1)]$.
h $= k/\alpha$.
g $= (k/\alpha)\cos\theta$.
w $= \exp[2\gamma(z-z_2)]$.
a $= k/2\gamma$.
b $= (k/2\gamma)\cos\theta$.
r =, r' = } reflexion coefficient for waves from {below, above} region I.
t =, t' = } transmission coefficient for waves from {below, above} region I.
ρ_{xx}, ρ_{yy} } = reflexion coefficients for region II (explained more fully below).
ρ_{xy}, β_{yx} } = conversion coefficients for region II (explained more fully below).
R_{xx}, R_{xy}, R_{yx}, R_{yy}, overall coefficients, explained below.

We shall take the positive z-axis as vertically upwards, and regard the ionosphere as horizontally stratified. We shall consider waves incident from below, with their wave normals in the x, z plane and making an angle θ with the vertical. If the incident wave is plane-polarized with its electric vector in the x, z plane, we shall find in general that in the reflected wave the electric vector is not confined entirely to this plane. We shall describe the ratio of the components of electric field in the x, z plane after and before reflexion, in terms of a 'reflexion coefficient' denoted by R_{xx} or ρ_{xx}, and the ratio of the component of the reflected electric field in the y, z plane to the incident electric field in the x, z plane will be described in terms of a 'conversion coefficient' denoted by R_{xy} or ρ_{xy}. If the incident wave is assumed to be linearly polarized with its electric field in the y, z plane, we shall describe the corresponding reflected wave in terms of similar reflexion coefficients R_{yy} or ρ_{yy}, and conversion coefficients R_{yx} or ρ_{yx}. The symbol ρ will apply to the reflexion from region II, and the symbol R to the composite wave produced below the ionosphere after reflexion from both regions together.

3. The two important regions

Wilkes showed that when the wave frequency was much less than the gyro-frequency p_H (the condition $|Y| \gg 1$), there are two separate levels in the ionosphere which play a decisive role in the propagation of the waves. He called one of these the 'transition region' and the other the 'reflecting region'; in this paper they are called region I and region II respectively. These two regions remain of particular significance in our theory.

Region I occurs where $X/|1-iZ|$ changes through unity from small to large values, but X/Y remains small compared with unity.

Region II is where $X/|1-iZ|\gg 1$, but X/Y is not negligible.

The theory will show that horizontally polarized waves are unaffected by passage through region I; but waves polarized with their electric field in the vertical plane are partially reflected and partially refracted when incident obliquely. If the theory of propagation in a 'slowly varying' medium is applied to this region and if Z is taken to be $\ll 1$, then the wave component with the electric field in the vertical plane would be totally reflected. Under certain conditions, e.g. at 16 kc/s, it will appear that X may increase so rapidly that the whole region is only a fraction of a wave-length thick. In this case an appreciable fraction of the energy leaks through.

The above inequalities, which govern the separation of the equations into two sets for the two regions, mean that in region II Z is negligible compared with $|Y|$. It will be shown that the wave in this region then breaks up into two elliptically polarized components, one of which is reflected completely, and the other only partially. The result is that, in general, the reflected wave has a component of electric field perpendicular to the plane of incidence, even if the incident wave is plane-polarized with its electric field entirely in the plane of incidence.

In the gap between the two regions, where $X/|1-iZ|\gg 1$ and $|X/Y|\ll 1$, two characteristic waves can be propagated, each with its own velocity. This will be more fully discussed in §10.

Under the above conditions the mathematical problem can conveniently be divided into two parts, one appropriate to each of the two regions. At a frequency of 16 kc/s, $|Y|\doteqdot 80$, and the treatment in terms of the two regions is undoubtedly valid. At a frequency of 80 kc/s, $|Y|\doteqdot 16$, and although we shall continue to work in terms of the two regions, we are near the range of frequency where this approximation is no longer valid.

4. The differential equations for region I

We take the axis of z to be vertical, the electron density (proportional to X) to vary only in the z direction, the imposed magnetic field to be along the z direction, and time variations to be given by $\exp(ikct)$. If a plane wave is incident from free space with its wave normal in the x, z plane making an angle θ with the z axis at $z=-\infty$, then all field vectors will vary like $\exp[ik(ct-x\sin\theta)]$. Under these conditions Wilkes (1947, equations (2·4) and (2·5)) showed that the field components satisfy the following equations:*

$$\frac{d^2}{dz^2}(E_x\pm iE_y)+\frac{d}{dz}\left(l\frac{dE_x}{dz}\right)+k^2\left(1-\frac{X}{1-iZ\mp Y}\right)(E_x\pm iE_y)\mp ik^2\sin^2\theta E_y=0, \qquad (4·1)$$

$$E_z=\frac{1}{ik\sin\theta}l\frac{dE_x}{dz}, \qquad (4·2)$$

where $$l=\sin^2\theta/[\cos^2\theta-X/(1-iZ)]. \qquad (4·3)$$

* Note that Wilkes used the factor $\exp[ik(ct+x\sin\theta)]$.

We now consider the form taken by these equations in region I. Here $X/|1-\mathrm{i}Z|$ is of the order unity, $|Y|\gg 1$ and $|Y|\gg X$. It follows also that $|1-\mathrm{i}Z\pm Y|\gg 1$, so that the term $X/(1-\mathrm{i}Z\mp Y)$ can be neglected.

Equations (4·1), (4·2) and (4·3) become

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1+l)\frac{\mathrm{d}E_x}{\mathrm{d}z}\right]+k^2E_x=0, \tag{4·4}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}E_y+k^2\cos^2\theta E_y=0, \tag{4·5}$$

$$E_z=\frac{1}{\mathrm{i}k\sin\theta}l\frac{\mathrm{d}E_x}{\mathrm{d}z}. \tag{4·6}$$

The wave represented by (4·5), which is polarized with its electric vector along the y-axis (horizontally polarized), is seen to travel at an angle θ to the z-axis, so that the wave suffers no refraction in region I.

The differential equation satisfied by E_z can be found by differentiating (4·4) and substituting for $\mathrm{d}E_x/\mathrm{d}z$ from (4·6). It is

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\left(\frac{1}{l}+1\right)E_z\right]+\frac{k^2}{l}E_z=0. \tag{4·7}$$

It is convenient first to solve this equation for E_z, then from equation (4·13) to deduce an expression for E_x, and hence calculate the reflexion and transmission coefficients.

Since $|X/(1-\mathrm{i}Z)|\ll 1$ below region I and $\gg 1$ above, equation (4·3) shows that l has the value $\tan^2\theta$ below and zero above. Equation (4·7) therefore shows that E_z varies as $\exp[-\mathrm{i}k(x\sin\theta+z\cos\theta)]$ below region I and as $\exp[-\mathrm{i}k(x\sin\theta+z)]$ above. A wave polarized with its electric field in the plane of propagation is hence refracted upwards in region I, and emerges with its wave normal making an angle ψ with the z-axis, where ψ is given by

$$\tan\psi=\sin\theta.$$

We now consider a particular model in which the electron density (proportional to X) increases exponentially upwards, and the collision frequency (proportional to Z) is constant, so that

$$X/(1-\mathrm{i}Z)=v(z)=\exp(\mathrm{i}\epsilon)\exp[\alpha(z-z_1)]. \tag{4·8}$$

In this expression $0\leqslant\epsilon<\frac{1}{2}\pi$. If $Z\ll 1$, then $\epsilon\to 0$; but in general, $\epsilon\neq 0$.

It is now convenient to make a change of independent variable from z to $v(z)$ in (4·7). This reduces it to an equation whose solution can be expressed in terms of hypergeometric functions. Substituting for $X/(1-\mathrm{i}Z)$ in (4·3) then gives

$$1+1/l=(1-v)/\sin^2\theta.$$

Also

$$\frac{\mathrm{d}E_z}{\mathrm{d}z}=\frac{\mathrm{d}E_z}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}z}=\alpha v\frac{\mathrm{d}E_z}{\mathrm{d}v}=\alpha\vartheta E_z,$$

where ϑ is the operator $v(\mathrm{d}/\mathrm{d}v)$. We can therefore rewrite (4·7) thus:

$$\alpha^2\vartheta^2[(1-v)E_z]+k^2(\cos^2\theta-v)E_z=0, \tag{4·9}$$

i.e.

$$\left(\vartheta^2+\frac{k^2}{\alpha^2}\cos^2\theta\right)E_z=\left(\vartheta^2+\frac{k^2}{\alpha^2}\right)(vE_z). \tag{4·10}$$

In (4·10) we now substitute
$$\left.\begin{aligned} g &= (k\cos\theta)/\alpha \\ h &= k/\alpha \end{aligned}\right\}, \tag{4·11}$$

to obtain
$$(\vartheta^2+g^2)\,E_z = (\vartheta^2+h^2)\,(vE_z); \tag{4·12}$$

the solutions of which will be investigated in the next section. Once E_z has been found, E_x can be obtained as follows:

$$k^2E_x = -\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{i}k\sin\theta(1/l+1)\,E_z\right]$$

$$= -\mathrm{i}k\sin\theta\,\alpha\vartheta\left(\frac{1-v}{\sin^2\theta}E_z\right),$$

i.e.
$$E_x = \frac{1}{\mathrm{i}h\sin\theta}\vartheta[(1-v)\,E_z]. \tag{4·13}$$

This expression is needed for the deduction of the reflexion and transmission coefficients in § 6. Strictly speaking, the treatment given here only applies to oblique incidence, since for normal incidence $E_z = 0$, and (4·13) is indeterminate as an expression for E_x. However, $E_z/\sin\theta$ remains finite as $\theta \to 0$; and although the solution could be carried out formally in terms of $E_z/\sin\theta$, this is not really necessary.

5. The fundamental solutions

By introducing the subsidiary change of dependent variable

$$E_z(v) = v^{\pm \mathrm{i}g}F_z(v)$$

into equation (4·12), we obtain for the equation satisfied by $F_z(v)$

$$\vartheta(\vartheta\pm 2\mathrm{i}g)\,F_z = v(\vartheta\pm\mathrm{i}g+\mathrm{i}h+1)\,(\vartheta\pm\mathrm{i}g-\mathrm{i}h+1)\,F_z.$$

This is the hypergeometric equation, having solutions (Copson 1935, § 10·3)

$$F_z(v) = {}_2F_1(\pm\mathrm{i}g+\mathrm{i}h+1,\ \pm\mathrm{i}g-\mathrm{i}h+1;\ \pm 2\mathrm{i}g+1;\ v).$$

Equation (4·12) thus has the two fundamental solutions

$$E_z^{(1)} = v^{\mathrm{i}g}\,{}_2F_1(\mathrm{i}g+\mathrm{i}h+1,\mathrm{i}g-\mathrm{i}h+1;\ 2\mathrm{i}g+1;\ v), \tag{5·1}$$

$$E_z^{(2)} = v^{-\mathrm{i}g}\,{}_2F_1(-\mathrm{i}g+\mathrm{i}h+1,\ -\mathrm{i}g-\mathrm{i}h+1;\ -2\mathrm{i}g+1;\ v). \tag{5·2}$$

Similarly there exists a second set, linearly dependent on (5·1) and (5·2):

$$E_z^{(3)} = v^{-\mathrm{i}h-1}\,{}_2F_1(\mathrm{i}h-\mathrm{i}g+1,\mathrm{i}h+\mathrm{i}g+1;\ 2\mathrm{i}h+1;\ 1/v), \tag{5·3}$$

$$E_z^{(4)} = v^{\mathrm{i}h-1}\,{}_2F_1(-\mathrm{i}h-\mathrm{i}g+1,\ -\mathrm{i}h+\mathrm{i}g+1;\ -2\mathrm{i}h+1;\ 1/v). \tag{5·4}$$

In the free space below the ionosphere, $|v| \ll 1$, and the hypergeometric functions in (5·1) and (5·2) are approximately unity. It follows that

$$E_z^{(1)} = v^{\mathrm{i}g} = \exp(-\epsilon g)\exp[\mathrm{i}\alpha g(z-z_1)]$$

$$= \exp(-\epsilon g)\exp[\mathrm{i}k\cos\theta(z-z_1)], \tag{5·5}$$

and, by substituting in (4·13), that

$$E_x^{(1)} = \frac{1}{\mathrm{i}h\sin\theta}\vartheta(v^{\mathrm{i}g}) = \frac{g}{h\sin\theta}v^{\mathrm{i}g}$$

$$= \cot\theta\exp(-\epsilon g)\exp[\mathrm{i}k\cos\theta(z-z_1)]. \tag{5·6}$$

Similarly from (5·2) it follows that

$$E_z^{(2)} = v^{-\mathrm{i}g} = \exp(\epsilon g)\exp[-\mathrm{i}k\cos\theta(z-z_1)] \tag{5·7}$$

and
$$E_x^{(2)} = -\cot\theta\exp(\epsilon g)\exp[-\mathrm{i}k\cos\theta(z-z_1)]. \tag{5·8}$$

It is now apparent that, below the ionosphere, where $|v| \ll 1$, affix 1 refers to a downcoming wave, and affix 2 to an upgoing wave.

Similarly, above region I, where $|v| \gg 1$, it can be shown that

$$E_z^{(3)} = v^{-\mathrm{i}h-1} = \exp(\epsilon h - \mathrm{i}\epsilon)\exp[-\alpha(z-z_1)]\exp[-\mathrm{i}k(z-z_1)], \tag{5·9}$$

$$E_x^{(3)} = \exp(\epsilon h)\exp[-\mathrm{i}k(z-z_1)]/\sin\theta, \tag{5·10}$$

and
$$E_z^{(4)} = v^{-\mathrm{i}h-1} = \exp(-\epsilon h - \mathrm{i}\epsilon)\exp[-\alpha(z-z_1)]\exp[\mathrm{i}k(z-z_1)], \tag{5·11}$$

$$E_x^{(4)} = -\exp(-\epsilon h)\exp[\mathrm{i}k(z-z_1)]/\sin\theta, \tag{5·12}$$

so that affix 3 refers to an upgoing wave and affix 4 to a downgoing wave.

6. Determination of the reflexion and transmission coefficients

Since equation (4·5) shows that the component E_y is unaffected by passage through region I, waves of the form $(E_x, 0, E_z)$ need alone be considered. When the incident wave is incident from below, the required boundary condition is that there is to be no wave propagated in the $-z$ direction for $|v| \gg 1$. Equations (5·11) and (5·12) show that $(E_x^{(4)}, 0, E_z^{(4)})$ is a wave propagated in the $-z$ direction for $|v| \gg 1$, and thus this solution is to be rejected. The wave represented by $(E_x^{(3)}, 0, E_z^{(3)})$ is the complete solution throughout the whole region, and satisfies the required boundary condition for $|v| \gg 1$. To derive from this solution the reflexion and transmission coefficients for waves incident from below, we follow an argument similar to that used by Epstein (1930). The solution $(E_x^{(3)}, 0, E_z^{(3)})$ is linearly dependent on the other set of fundamental solutions (5·1) and (5·2), which, for $|v| \ll 1$, represent, respectively, the reflected and incident waves below the layer (equations (5·5), (5·7)). In order to obtain the relative proportions of the incident, reflected and transmitted waves, we must connect the hypergeometric functions of argument $1/v$ with those of argument v. Following Copson (1935, § 10·41), if $|\arg(-v)| < \pi$, the required relation is

$$ {}_2F_1(a,b;c;v) = \frac{(c-1)!\,(b-a-1)!}{(c-a-1)!\,(b-1)!}(-v)^{-a}\,{}_2F_1(a,1-c+a;1-b+a;1/v)$$
$$+\frac{(c-1)!\,(a-b-1)!}{(c-b-1)!\,(a-1)!}(-v)^{-b}\,{}_2F_1(b,1-c+b;1-a+b;1/v).$$

Now $0 < \arg v = \epsilon < \frac{1}{2}\pi$, and hence $-v$ must be interpreted as $v\exp(-\mathrm{i}\pi)$. Applying this formula to (5·3),

$$E_z^{(3)} = -\exp[\pi(h-g)]\frac{(2\mathrm{i}h)!\,(2\mathrm{i}g-1)!}{(\mathrm{i}h+\mathrm{i}g)!\,(\mathrm{i}h+\mathrm{i}g-1)!}E_z^{(2)} - \exp[\pi(h+g)]\frac{(2\mathrm{i}h)!\,(-2\mathrm{i}g-1)!}{(\mathrm{i}h-\mathrm{i}g)!\,(\mathrm{i}h-\mathrm{i}g-1)!}E_z^{(1)}.$$

From formula (4·13), it follows that the same linear relation holds between $E_x^{(1)}$, $E_x^{(2)}$ and $E_x^{(3)}$. Writing $E_x^{(3)}$ as E_x, the field throughout the region, formulae (5·6) and (5·8) can be used for $E_x^{(1)}$ and $E_x^{(2)}$, so that when $z \ll z_1$, i.e. $|v| \ll 1$,

$$E_x \sim \exp[\pi(h-g)] \frac{(2ih)!\,(2ig-1)!}{(ih+ig)!\,(ih+ig-1)!} \cot\theta \exp(\epsilon g) \exp[-ik\cos\theta(z-z_1)]$$

$$-\exp[\pi(h+g)] \frac{(2ih)!\,(-2ig-1)!}{(ih-ig)!\,(ih-ig-1)!} \cot\theta \exp(-\epsilon g) \exp[ik\cos\theta(z-z_1)]. \quad (6·1)$$

Similarly, above the region, where $z \gg 1$, i.e. $|v| \gg 1$, using (5·10)

$$E_x \sim \exp(\epsilon h) \exp[-ik(z-z_1)]/\sin\theta. \quad (6·2)$$

Now take $C\cos\theta \exp[-ik\cos\theta(z-z_1)]$ as the incident field,

$Cr\cos\theta \exp[ik\cos\theta(z-z_1)]$ as the reflected field,

$Ct\exp[-ik(z-z_1)]$ as the transmitted field,

where C is a constant, and r and t are the reflexion and transmission coefficients respectively of region I for waves incident from below. Comparing the coefficients C, Cr, Ct with those in (6·1) and (6·2), we obtain the results:

$$r = -\exp[2g(\pi-\epsilon)] \frac{(ih+ig-1)!\,(ih+ig)!\,(-2ig-1)!}{(ih-ig-1)!\,(ih-ig)!\,(2ig-1)!}$$

$$= \frac{1-\cos\theta}{1+\cos\theta} \exp[2g(\pi-\epsilon)] \frac{(-2ig)!}{(2ig)!} \left[\frac{(ih+ig)!}{(ih-ig)!}\right]^2, \quad (6·3)$$

$$t = \exp[-(h-g)(\pi-\epsilon)] \frac{(ih+ig-1)!\,(ih+ig)!}{(2ih)!\,(2ig-1)!}$$

$$= \frac{2\cos\theta}{1+\cos\theta} \exp[-(h-g)(\pi-\epsilon)] \frac{[(ih+ig)!]^2}{(2ih)!\,(2ig)!}. \quad (6·4)$$

If the gradient of ionization is great, so that α (equation (4·8)) is large, then g and h (equations (4·11)) are small and the factorial functions in equations (6·3) and (6·4) tend to unity, so that the reflexion and transmission coefficients take the simple form

$$r = (1-\cos\theta)/(1+\cos\theta)$$

$$t = 2\cos\theta/(1+\cos\theta).$$

In order to investigate the behaviour of waves incident on region I from above, the required boundary condition is that the solution $E_x^{(2)}$ is omitted, and $E_x^{(1)}$ is taken to represent the wave throughout the medium. $E_x^{(1)}$ is expressed in terms of $E_x^{(3)}$ and $E_x^{(4)}$, and a treatment similar to that just given leads to the following values for the reflexion coefficient r' and the transmission coefficient t':

$$r' = \exp[-2h(\pi-\epsilon)] \frac{(ig+ih-1)!\,(ig+ih)!\,(-2ih-1)!}{(ig-ih-1)!\,(ig-ih)!\,(2ih-1)!}$$

$$= -\frac{1-\cos\theta}{1+\cos\theta} \exp[-2h(\pi-\epsilon)] \frac{(-2ih)!}{(2ih)!} \left[\frac{(ig+ih)!}{(ig-ih)!}\right]^2, \quad (6·5)$$

$$t' = \exp[-(h-g)(\pi-\epsilon)] \frac{(ig+ih-1)!\,(ig+ih)!}{(2ig)!\,(2ih-1)!}$$

$$= \frac{2}{1+\cos\theta} \exp[-(h-g)(\pi-\epsilon)] \frac{[(ig+ih)!]^2}{(2ig)!\,(2ih)!}. \quad (6·6)$$

We notice that
$$t' = t \sec\theta. \tag{6·7}$$
The moduli $|r|$, $|t|$, $|r'|$, $|t'|$, can be found by applying the formulae
$$(z-1)!\,(-z)! = \pi \operatorname{cosec}(\pi z),$$
$$|(\mathrm{i}g+\mathrm{i}h)!| = |(-\mathrm{i}g-\mathrm{i}h)!|$$
to equations (6·3) to (6·6). The moduli are given by
$$\begin{aligned}|r| &= \exp[2g(\pi-\epsilon)]\frac{\sinh\pi(h-g)}{\sinh\pi(h+g)}\\ &= \exp(-2g\epsilon)\frac{1-\exp[-2\pi(h-g)]}{1-\exp[-2\pi(h+g)]},\end{aligned} \tag{6·8}$$
$$\begin{aligned}|t| &= \surd(\cos\theta)\exp[-(h-g)(\pi-\epsilon)]\frac{\surd(\sinh 2\pi h\sinh 2\pi g)}{\sinh\pi(h+g)}\\ &= \surd(\cos\theta)\exp[-(h-g)(\pi-\epsilon)]\frac{\surd\{[1-\exp(-4\pi h)][1-\exp(-4\pi g)]\}}{1-\exp[-2\pi(h+g)]},\end{aligned} \tag{6·9}$$
$$\begin{aligned}|r'| &= \exp[-2h(\pi-\epsilon)]\frac{\sinh\pi(h-g)}{\sinh\pi(h+g)}\\ &= \exp(2h\epsilon)\exp[-2(h+g)\pi]\frac{1-\exp[-2\pi(h-g)]}{1-\exp[-2\pi(h+g)]},\end{aligned} \tag{6·10}$$
$$|t'| = |t|\sec\theta. \tag{6·11}$$
The phases, referred to the level z_1, are given by
$$\arg r = -2\arg(2\mathrm{i}g)! + 2\arg(\mathrm{i}h+\mathrm{i}g)! - 2\arg(\mathrm{i}h-\mathrm{i}g)!, \tag{6·12}$$
$$\arg t = 2\arg(\mathrm{i}h+\mathrm{i}g)! - \arg(2\mathrm{i}h)! - \arg(2\mathrm{i}g)!, \tag{6·13}$$
$$\arg r' = \pi - 2\arg(2\mathrm{i}h)! + 2\arg(\mathrm{i}h+\mathrm{i}g)! + 2\arg(\mathrm{i}h-\mathrm{i}g)!, \tag{6·14}$$
$$\arg t' = \arg t. \tag{6·15}$$

In interpreting these expressions, it should be remembered that $h = k/\alpha = 2\pi/(\lambda_0\alpha)$, where α is given by equation (4·8) and $g = (k/\alpha)\cos\theta$. The expressions (6·8) to (6·15) for the modulus and argument of the reflexion and transmission coefficients of region I represent the effect of this region on the vertically polarized part of the wave. They are combined in § 9 with corresponding coefficients showing the effect of region II.

A peculiar consequence of equations (6·8) to (6·11) should be noticed. Suppose that $\epsilon \to 0$, whilst $g\pi$, $h\pi$ and $(h-g)\pi \gg 1$. The first condition implies that Z, and therefore ν, are small, so we would not expect much dissipation of energy, i.e. absorption. The other conditions imply that the change of ionization per wave-length is small, and the wave is incident obliquely. Then from equations (6·8) to (6·11)
$$|r| \to 1, \quad |t| \to 0, \quad |r'| \to 0, \quad |t'| \to 0. \tag{6·16}$$

The first two of these relations show that a wave incident from below is almost perfectly reflected, as would be expected, but the last two relations show that a wave incident from above is strongly absorbed, which seems inconsistent. The explanation of this result will be given in appendix 3.

7. The differential equations for region II

We define region II as being the region in which

$$|Y| \gg Z, \quad |Y| \gg 1$$

and $|X/Y|$ of order unity.

The equations (4·1) to (4·3) become

$$\frac{d^2E_x}{dz^2}+k^2E_x = -ik^2\frac{X}{Y}E_y, \tag{7·1}$$

$$\frac{d^2E_y}{dz^2}+k^2\cos^2\theta E_y = ik^2\frac{X}{Y}E_x. \tag{7·2}$$

We observe that since l is now small, E_z is negligible. As before, a special model ionosphere is considered in which the electron density (proportional to X) increases upwards exponentially with height. It is not necessary to assume, however, that the rate of exponential increase is the same as in region I. We therefore write

$$\frac{X}{Y} = \pm\exp[\gamma(z-z_3)],$$

where z_3 is the height where $|X/Y| = 1$, and where the two signs are included to allow for the different signs of Y in the southern and northern hemisphere. Throughout this treatment, the lower sign will refer consistently to the northern hemisphere. In order to simplify the algebra, it is advantageous to work in terms of a reference height z_2, rather than z_3, where

$$z_2 = z_3+\frac{2}{\gamma}\log_e(2\gamma/k).$$

Thus

$$\frac{X}{Y} = \pm\frac{4\gamma^2}{k^2}\exp[\gamma(z-z_2)]. \tag{7·3}$$

As in the case of region I, it is convenient to make a change of independent variable by writing

$$w = \exp[2\gamma(z-z_2)]. \tag{7·4}$$

Let

$$\left.\begin{aligned} a &= k/2\gamma, \\ b &= (k\cos\theta)/2\gamma. \end{aligned}\right\} \tag{7·5}$$

Then

$$\frac{dE}{dz} = \frac{dE}{dw}\,\frac{dw}{dz} = 2\gamma w\frac{dE}{dw} = 2\gamma\vartheta E,$$

where ϑ is the operator $w(d/dw)$. In terms of the new independent variable w, (7·1) and (7·2) become respectively

$$(\vartheta^2+a^2)E_x = \mp i\surd(w)E_y \tag{7·6}$$

and

$$(\vartheta^2+b^2)E_y = \pm i\surd(w)E_x. \tag{7·7}$$

To eliminate E_y from these equations, we notice that

$$\vartheta[\surd(w)E_y] = E_y\vartheta\surd(w)+\surd(w)\vartheta E_y = \surd(w)(\vartheta+\tfrac{1}{2})E_y,$$

or

$$\surd(w)\vartheta E_y = (\vartheta-\tfrac{1}{2})[\surd(w)E_y].$$

Thus

$$\surd(w)\vartheta^2E_y = (\vartheta-\tfrac{1}{2})^2[\surd(w)E_y].$$

Application of this last equation to (7·7) gives

$$\begin{aligned} wE_x &= \mp i\sqrt{(w)}\,(\vartheta^2+b^2)\,E_y \\ &= [(\vartheta-\tfrac{1}{2})^2+b^2]\,[\mp i\sqrt{(w)}\,E_y] \\ &= [(\vartheta-\tfrac{1}{2})^2+b^2]\,(\vartheta^2+a^2)\,E_x, \end{aligned} \tag{7·8}$$

and similarly $$wE_y = [(\vartheta-\tfrac{1}{2})^2+a^2]\,(\vartheta^2+b^2)\,E_y. \tag{7·9}$$

These two equations are of the general form

$$LF \equiv (\vartheta-p_1)\,(\vartheta-p_2)\,(\vartheta-p_3)\,(\vartheta-p_4)\,F = wF, \tag{7·10}$$

where L is the operator $(\vartheta-p_1)\,(\vartheta-p_2)\,(\vartheta-p_3)\,(\vartheta-p_4)$; and solutions of this equation will be investigated in the next section.

Equation (7·10) is a generalization of Bessel's differential equation. If we substitute $z = 2i\sqrt{w}$ in Bessel's equation

$$\frac{d^2F}{dz^2}+\frac{1}{z}\frac{dF}{dz}+\left(1-\frac{n^2}{z^2}\right)F = 0,$$

we obtain $$(\vartheta^2-\tfrac{1}{4}n^2)\,F = wF, \tag{7·11}$$

which is similar to (7·10), but only of the second order. One fundamental set of solutions of (7·11) is $J_n(2i\sqrt{w})$ and $J_{-n}(2i\sqrt{w})$ if n is not an integer. It also has a second fundamental set of solutions, the two Hankel functions $H_n^{(1)}(2i\sqrt{w})$ and $H_n^{(2)}(2i\sqrt{w})$; and because

$$H_n^{(1)}(z\,e^{\pi i}) = -e^{-n\pi i}\,H_n^{(2)}(z),$$

this second set may be taken as $H_n^{(1)}(2i\sqrt{w})$ and $H_n^{(1)}(2i\sqrt{w}\,e^{i\pi})$. These Hankel functions have simple asymptotic forms when the argument is large, and any boundary conditions to be imposed for $|w|$ large can be applied to these asymptotic forms.

Considerations such as these can be extended to the generalized equation (7·10). Being of the fourth order, there are four independent solutions, and by taking linearly independent combinations of these four solutions other sets of four independent solutions can be formed. § 8 is concerned with three such sets. The first set is given by the four power series of (8·2), and these represent, as shown in § 9, the incident and reflected waves. The solutions of the second set are given by the Barnes integral of (8·3) for various contours, and each are shown by (8·4) to be respectively proportional to the set (8·2). The third set, given by (8·5) for a further contour for the Barnes integral, is expressed as a linear sum of the four solutions given by (8·4) (this third set is analogous to the Hankel functions). The boundary conditions for large $|w|$ can only be applied to this third set, and simple asymptotic forms for this third set determine which of the four solutions satisfy the boundary conditions and which do not. The reflexion coefficients are then found, in § 9, from the two solutions which satisfy the required boundary conditions. In § 9 also, the coupling existing between the equations (7·8) and (7·9) is taken into account.

8. Solutions of the differential equation (7·10)

The equation $$LF \equiv (\vartheta-p_1)\,(\vartheta-p_2)\,(\vartheta-p_3)\,(\vartheta-p_4)\,F = wF \tag{8·1}$$

is a special case of the generalized hypergeometric equation, treated by Copson (1935, § 10·6).

The equation can be solved in terms of four power series of the type

$$P_1(w) = w^{p_1} \sum_{n=0}^{\infty} \frac{(p_1-p_2)!\,(p_1-p_3)!\,(p_1-p_4)!}{(p_1-p_2+n)!\,(p_1-p_3+n)!\,(p_1-p_4+n)!}\frac{w^n}{n!}. \tag{8·2}$$

This sum is a generalized hypergeometric function, and written in standard notation the solution is

$$P_1(w) = w^{p_1}\,{}_0F_3(p_1-p_2+1, p_1-p_3+1, p_1-p_4+1; w),$$

together with three similar solutions.

To investigate the relation between the solutions, (8·1) may also be solved in terms of Barnes's integrals. Consider, for example, the Barnes integral

$$I_m(w) = \frac{1}{2\pi \mathrm{i}}\int_{C_m} (s+p_1-1)!\,(s+p_2-1)!\,(s+p_3-1)!\,(s+p_4-1)!\,w^{-s}\,\mathrm{d}s$$

$$= \frac{1}{2\pi \mathrm{i}}\int_{C_m} \prod_{j=1}^{4} (s+p_j-1)!\,w^{-s}\,\mathrm{d}s, \tag{8·3}$$

where C_m is one of the paths of figure 1 starting and finishing at infinity in the negative half-plane. It can easily be shown that $I(w)$ satisfies the differential equation (8·1); this will, in fact, be shown by (8·4). The poles of the integrand are at the points

$$\left.\begin{aligned} s &= -p_1-n\\ s &= -p_2-n\\ s &= -p_3-n\\ s &= -p_4-n\end{aligned}\right\}\; n \text{ zero, or a positive integer.}$$

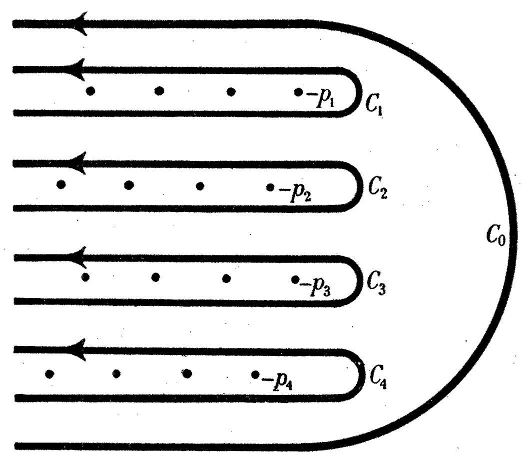


FIGURE 1. Contours in the s-plane for the Barnes integral 8·3.

Consider the five paths $C = C_0, C_1, C_2, C_3, C_4$ as in figure 1, and denote the corresponding integrals by I_0, I_1, I_2, I_3, I_4 respectively. Then

$$I_0 = I_1+I_2+I_3+I_4.$$

$I_1(w) =$ sum of residues of $\prod_{j=1}^{4}(s+p_j-1)!\,w^{-s}$ at $s=-p_1-n$.

The residues are

$$\frac{(-p_1-n+p_2-1)!\,(-p_1-n+p_3-1)!\,(-p_1-n+p_4-1)!}{(-)^{n+1}\,n!}\,w^{p_1+n}.$$

This yields $$I_1(w) = -(p_2-p_1-1)!\,(p_3-p_1-1)!\,(p_4-p_1-1)!\,P_1(w), \tag{8·4}$$
where $P_1(w)$ is defined in (8·2). Equation (8·4), together with similar ones for I_2, I_3, I_4 verify that $I(w)$ satisfies (8·1).

We need four independent solutions of (8·1) for the complete solution. The equation $LF = wF$ is also satisfied by $I_0(w\,e^{2i\pi n})$. In particular, taking $n = 0$, 1, -1 and 2, the four functions

$$\left.\begin{aligned} I_0(w) &= I_1(w) + I_2(w) + I_3(w) + I_4(w),\\ I_0(w\,e^{2i\pi}) &= \Omega_1 I_1(w) + \Omega_2 I_2(w) + \Omega_3 I_3(w) + \Omega_4 I_4(w),\\ I_0(w\,e^{-2i\pi}) &= \Omega_1^{-1} I_1(w) + \Omega_2^{-1} I_2(w) + \Omega_3^{-1} I_3(w) + \Omega_4^{-1} I_4(w),\\ I_0(w\,e^{4i\pi}) &= \Omega_1^2 I_1(w) + \Omega_2^2 I_2(w) + \Omega_3^2 I_4(w) + \Omega_4^2 I_4(w), \end{aligned}\right\} \tag{8·5}$$

form a new fundamental set, where

$$\Omega_j = \exp(2i\pi p_j).$$

When $|w|$ is large, the integral $I_0(w)$ may be evaluated asymptotically by the method of steepest descents, using Stirling's formula as an approximation to the factorial functions. The analysis, which is somewhat involved, is relegated to an appendix.

The results are as follows:

If (a) $|w| \gg \text{Max}[1, |p_1|, |p_2|, |p_3|, |p_4|]$,

(b) $|\arg w| \leqslant 4\pi$,

$$I_0(w) \sim \sqrt{2}\,\pi^{\frac{3}{2}} w^{-\frac{3}{8}+\Sigma_j p_j/4} \exp(-4w^{\frac{1}{4}}) \exp[O(w^{-\frac{1}{4}})].$$

For $|\arg w| < 2\pi$, the real part of $(-4w^{\frac{1}{4}})$ is negative, so $|I_0(w)|$ decreases exponentially with w; while for $|\arg w| > 2\pi$, the real part of $(-4w^{\frac{1}{4}})$ is positive, so $|I_0(w)|$ increases exponentially.

The boundary conditions required for region II are that for waves which penetrate the layer, i.e. for large $|w|$, the modulus of I_0 should decrease, and that the direction of propagation should be upwards. We must pick solutions from (8·5) which satisfy these conditions; for which the dominant factor in the asymptotic form is $\exp(-4w^{\frac{1}{4}})$. For the four solutions (8·5), this factor respectively takes the form

(a) $\exp(-4w^{\frac{1}{4}})$,

(b) $\exp(-4iw^{\frac{1}{4}})$,

(c) $\exp(4iw^{\frac{1}{4}})$,

(d) $\exp(4w^{\frac{1}{4}})$,

where w is real.

(a) decreases upwards exponentially, and is consistent with the upper boundary conditions.

(b) represents a wave propagated upwards, and is consistent with the upper boundary conditions.

(c) represents a wave propagated downwards, and must be rejected.

(d) increases upwards exponentially, and must be rejected.

The complete solution satisfying the upper boundary conditions is thus a linear combination of (*a*) and (*b*), namely,

$$I_0(w)+AI_0(w\,e^{2i\pi}), \tag{8·6}$$

where A is a constant; and this combination is taken as representing F throughout the region.

9. Reflexion and conversion coefficients for region II

We will now apply the general formulae obtained in the last section to (7·8) and (7·9). Denote the p's referring to (7·8) by $p^{(x)}$, and those referring to (7·9) by $p^{(y)}$. Similarly, denote the various solutions of (7·8) by $I^{(x)}(w)$, and of (7·9) by $I^{(y)}(w)$.

Then for E_x, from (7·8)

$$p_1^{(x)}=\mathrm{i}a,$$
$$p_2^{(x)}=-\mathrm{i}a,$$
$$p_3^{(x)}=\mathrm{i}b+\tfrac{1}{2},$$
$$p_4^{(x)}=-\mathrm{i}b+\tfrac{1}{2},$$

and for E_y, from (7·9)

$$p_1^{(y)}=\mathrm{i}a+\tfrac{1}{2},$$
$$p_2^{(y)}=-\mathrm{i}a+\tfrac{1}{2},$$
$$p_3^{(y)}=\mathrm{i}b,$$
$$p_4^{(y)}=-\mathrm{i}b.$$

It is necessary to see how the solutions behave below region II, that is, as

$$w\to 0,$$

or

$$z-z_2=(\log_e w)/2\gamma\to-\infty,\quad \text{from (7·4)}.$$

From (8·2) and (8·4)

$$I_1(w)\propto P_1(w)\sim w^{p_1}=\exp\left[2\gamma p_1(z-z_2)\right].$$

Then

$$I_1^{(x)}(w)\propto w^{\mathrm{i}a}=\exp\left[2\gamma \mathrm{i}a(z-z_2)\right]=\exp\left[\mathrm{i}k(z-z_2)\right],$$
$$I_2^{(x)}(w)\propto w^{-\mathrm{i}a}=\exp\left[-2\gamma \mathrm{i}a(z-z_2)\right]=\exp\left[-\mathrm{i}k(z-z_2)\right],$$
$$I_3^{(x)}(w)\propto w^{\mathrm{i}b+\frac{1}{2}}=\exp\left[2\gamma(\mathrm{i}b+\tfrac{1}{2})\,(z-z_2)\right]=\exp\left[\gamma(z-z_2)\right]\exp\left[\mathrm{i}k(z-z_2)\cos\theta\right],$$
$$I_4^{(x)}(w)\propto w^{-\mathrm{i}b+\frac{1}{2}}=\exp\left[2\gamma(-\mathrm{i}b+\tfrac{1}{2})\,(z-z_2)\right]=\exp\left[\gamma(z-z_2)\right]\exp\left[-\mathrm{i}k(z-z_2)\cos\theta\right],$$

and

$$I_1^{(y)}(w)\propto w^{\mathrm{i}a+\frac{1}{2}}=\exp\left[2\gamma(\mathrm{i}a+\tfrac{1}{2})\,(z-z_2)\right]=\exp\left[\gamma(z-z_2)\right]\exp\left[\mathrm{i}k(z-z_2)\right],$$
$$I_2^{(y)}(w)\propto w^{-\mathrm{i}a+\frac{1}{2}}=\exp\left[2\gamma(-\mathrm{i}a+\tfrac{1}{2})\,(z-z_2)\right]=\exp\left[\gamma(z-z_2)\right]\exp\left[-\mathrm{i}k(z-z_2)\right],$$
$$I_3^{(y)}(w)\propto w^{\mathrm{i}b}=\exp\left[2\gamma \mathrm{i}b(z-z_2)\right]=\exp\left[\mathrm{i}k(z-z_2)\cos\theta\right],$$
$$I_4^{(y)}(w)\propto w^{-\mathrm{i}b}=\exp\left[-2\gamma \mathrm{i}b(z-z_2)\right]=\exp\left[-\mathrm{i}k(z-z_2)\cos\theta\right].$$

Thus for $z\ll z_2$, $I_3^{(x)}(w)$, $I_4^{(x)}(w)$, $I_1^{(y)}(w)$, $I_2^{(y)}(w)$ are negligible, and

$I_1^{(x)}$ represents the reflected E_x field, $E_x^{(r)}$,

$I_2^{(x)}$ represents the incident E_x field, $E_x^{(i)}$,

$I_3^{(y)}$ represents the reflected E_y field, $E_y^{(r)}$,

$I_4^{(y)}$ represents the incident E_y field, $E_y^{(i)}$.

The connexion between the corresponding solutions for E_x and E_y is easily obtained in the following way from the representation of the solutions in terms of the Barnes integrals. From (8·3)

$$E_x = I_m^{(x)}(w\,e^{2\pi n i}) = \frac{1}{2\pi i}\int_{C_m}\prod_{j=1}^{4}(s+p_j^{(x)}-1)!\,w^{-s}\,e^{-2\pi n i s}\,ds,$$

where $m = 1, 2, 3, 4$. E_y is then obtained from (7·6),

$$(\vartheta^2+a^2)\,E_x = \mp i\sqrt{(w)}\,E_y,$$

i.e.

$$\mp i\sqrt{(w)}\,E_y = \frac{1}{2\pi i}(\vartheta^2+a^2)\int_{C_m}\prod_{j=1}^{4}(s+p_j^{(x)}-1)!\,w^{-s}\,e^{-2\pi n i s}\,ds$$
$$= \frac{1}{2\pi i}\int_{C_m}\prod_{j=1}^{4}(s+p_j^{(x)}-1)!\,(s^2+a^2)\,w^{-s}\,e^{-2\pi n i s}\,ds,$$

because $\vartheta = w\,d/dw$ operates only on the factor w^{-s}:

$$\mp i\sqrt{(w)}\,E_y = \frac{1}{2\pi i}\int_{C_m}(s+p_1^{(x)})!\,(s+p_2^{(x)})!\,(s+p_3^{(x)}-1)!\,(s+p_4^{(x)}-1)!\,w^{-s}\,e^{-2\pi n i s}\,ds,$$

because

$$s^2+a^2 = (s+ia)\,(s-ia) = (s+p_1^{(x)})\,(s+p_2^{(x)}),$$

$$\mp i\sqrt{(w)}\,E_y = \frac{1}{2\pi i}\int_{C_m}\prod_{j=1}^{4}(t+p_j^{(y)}-1)!\,w^{-t}\sqrt{(w)}\,e^{-2\pi n i t}\,e^{\pi n i}\,dt,$$

where $s+\frac{1}{2} = t$, not affecting the contour C_m. Thus

$$\mp i\sqrt{(w)}\,E_y = \sqrt{(w)}\,e^{\pi n i}\,I_m^{(y)}(w\,e^{2\pi n i}).$$

Therefore

$$E_y = \pm i\,e^{-in\pi}\,I_m^{(y)}(w\,e^{2\pi n i}),$$

when

$$E_x = I_m^{(x)}(w\,e^{2\pi n i}).$$

It follows that $I_3^{(x)}(w)$ is associated with $\pm i I_3^{(y)}(w)$,
$I_4^{(x)}(w)$ is associated with $\pm i I_4^{(y)}(w)$,
$I_3^{(x)}(w\,e^{2\pi i})$ is associated with $\mp i\exp(2\pi i p_3^{(y)})\,I_3^{(y)}(w)$,
$I_4^{(x)}(w\,e^{2\pi i})$ is associated with $\mp i\exp(2\pi i p_4^{(y)})\,I_4^{(y)}(w)$.

Applying (8·6), the waves that contribute to the incident and reflected fields are in the following proportions:

$$[1+A\exp(2\pi i p_1^{(x)})]\,I_1^{(x)}(w) : [1+A\exp(2\pi i p_2^{(x)})]\,I_2^{(x)}(w)$$
$$: \pm i[1-A\exp(2\pi i p_3^{(y)})]\,I_3^{(y)}(w) : \pm i[1-A\exp(2\pi i p_4^{(y)})]\,I_4^{(y)}(w),$$

which for $|w|$ small give the proportions of

$$E_x^{(r)} : E_x^{(i)} : E_y^{(r)} : E_y^{(i)},$$

namely, from (8·4)

$$[1+A\exp(-2\pi a)]\,(-2ia-1)!\,(-\tfrac{1}{2}+ib-ia)!\,(-\tfrac{1}{2}-ib-ia)!\exp[ik(z-z_2)]$$
$$:[1+A\exp(2\pi a)]\,(2ia-1)!\,(-\tfrac{1}{2}+ib+ia)!\,(-\tfrac{1}{2}-ib+ia)!\exp[-ik(z-z_2)]$$
$$:\pm i[1-A\exp(-2\pi b)]\,(-2ib-1)!\,(-\tfrac{1}{2}+ia-ib)!\,(-\tfrac{1}{2}-ia-ib)!\exp[ik(z-z_2)\cos\theta]$$
$$:\pm i[1-A\exp(2\pi b)]\,(2ib-1)!\,(-\tfrac{1}{2}+ia+ib)!\,(-\tfrac{1}{2}-ia+ib)!\exp[-ik(z-z_2)\cos\theta].$$

From this set of ratios we can see the individual contributions from $I_0(w)$ and $I_0(w\,e^{2\pi i})$. By taking their moduli, it can be seen that $I_0(w)$ represents an elliptically polarized wave, with the sense of polarization maintained on reflexion. This wave is totally reflected, as the layer is non-absorbing and no energy is being transmitted. This is analogous to the ordinary component in the magneto-ionic theory. Similarly, $I_0(w\,e^{2\pi i})$ represents an elliptically polarized wave; but in this mode there is only partial reflexion, because as we saw at the end of § 8, the asymptotic form of $I_0(w\,e^{2\pi i})$ for large $|w|$ represents a wave propagated upwards, meaning a partial penetration of the wave. This corresponds to the extraordinary component in the magneto-ionic theory.

It is the purpose of this paper to deal with the plane-polarized components of the wave, and we now introduce the reflexion and conversion coefficients $\rho_{xx}, \rho_{xy}, \rho_{yx}, \rho_{yy}$ defined in § 2. If $E_y^{(i)}=0$, $A=\exp(-2\pi b)$. Hence

$$\rho_{xx}=\frac{1+\exp[-2\pi(a+b)]}{1+\exp[2\pi(a-b)]}\,\frac{(-2ia-1)!\,(-\frac{1}{2}-ia+ib)!\,(-\frac{1}{2}-ia-ib)!}{(2ia-1)!\,(-\frac{1}{2}+ia+ib)!\,(-\frac{1}{2}+ia-ib)!},$$

$$\rho_{xy}=\frac{\pm i[1-\exp(-4\pi b)]}{1+\exp[2\pi(a-b)]}\,\frac{(-2ib-1)!\,(-\frac{1}{2}-ia-ib)!}{(2ia-1)!\,(-\frac{1}{2}+ia+ib)!}.$$

If $E_x^{(i)}=0$, $A=-\exp(-2\pi a)$. Hence

$$\rho_{yx}=\frac{1-\exp(-4\pi a)}{\pm i\{1+\exp[2\pi(b-a)]\}}\,\frac{(-2ia-1)!\,(-\frac{1}{2}-ia-ib)!}{(2ib-1)!\,(-\frac{1}{2}+ia+ib)!},$$

$$\rho_{yy}=\frac{1+\exp[-2\pi(a+b)]}{1+\exp[-2\pi(a-b)]}\,\frac{(-2ib-1)!\,(-\frac{1}{2}+ia-ib)!\,(-\frac{1}{2}-ia-ib)!}{(2ib-1)!\,(-\frac{1}{2}+ia+ib)!\,(-\frac{1}{2}-ia+ib)!}.$$

The moduli and phases referred to the level z_2 are

$$|\rho_{xx}|=\exp(-2\pi a)\frac{\cosh\pi(a+b)}{\cosh\pi(a-b)}=\exp[-2\pi(a-b)]\frac{1+\exp[-2\pi(a+b)]}{1+\exp[-2\pi(a-b)]}, \tag{9·1}$$

$$|\rho_{xy}|=\frac{\exp[-\pi(a+b)]}{\sqrt{\cos\theta}}\,\frac{\sqrt{(\sinh 2\pi a\,\sinh 2\pi b)}}{\cosh\pi(a-b)}$$

$$=\frac{\exp[-\pi(a-b)]}{\sqrt{\cos\theta}}\,\frac{\sqrt{\{[1-\exp(-4\pi a)]\,[1-\exp(-4\pi b)]\}}}{1+\exp[2\pi(a-b)]}, \tag{9·2}$$

$$|\rho_{yx}|=|\rho_{xy}|\cos\theta, \tag{9·3}$$

$$|\rho_{yy}|=\exp(-2\pi b)\frac{\cosh\pi(a+b)}{\cosh\pi(a-b)}=\frac{1+\exp[-2\pi(a+b)]}{1+\exp[-2\pi(a-b)]}, \tag{9·4}$$

$$\arg\rho_{xx}=\pi-2\arg(2ia)!-2\arg(ia-ib-\tfrac{1}{2})!-2\arg(ia+ib-\tfrac{1}{2})!, \tag{9·5}$$

$$\arg\rho_{xy}=\mp\frac{\pi}{2}-\arg(2ia)!-\arg(2ib)!-2\arg(ia+ib-\tfrac{1}{2})!, \tag{9·6}$$

$$\arg\rho_{yx}=\pm\pi+\arg\rho_{xy}, \tag{9·7}$$

$$\arg\rho_{yy}=\pi-2\arg(2ib)!+2\arg(ia-ib-\tfrac{1}{2})!-2\arg(ia+ib-\tfrac{1}{2})!. \tag{9·8}$$

The expressions (9·1) to (9·8) giving the modulus and argument of the reflexion and conversion coefficients suffice to describe the effect of region II on the wave. In interpreting them, it should be remembered that $a = k/2\gamma$, $b = (k/2\gamma)\cos\theta$, where γ is given by equation (7·3).

10. The combined effect of the two regions, and the deduction of the overall reflexion coefficients

Consider now what happens when a plane wave is incident from below on an ionosphere of the type hitherto considered; that is, regarded as containing two separated regions having the properties stated in § 3. In each region separately, the ionization density is taken to increase exponentially with height, though not necessarily with the same exponent, so that the transmission and reflexion coefficients of the regions have the values deduced in the previous sections.

We shall also require to know how the wave is propagated in the space between regions I and II. This space is characterized by the conditions $X/|1-\mathrm{i}Z| \gg 1$ and $|X/Y| \ll 1$. The propagation equations are the limiting forms taken by the equations for region I towards the top of the region, that is, for $X/|1-\mathrm{i}Z|$ large; and these forms are the same as assumed by the equations for region II towards the base of the region, that is, for $|X/Y|$ small. The forms taken by (7·1) and (7·2) below region II, where $|X/Y|$ is small, are

$$\frac{\mathrm{d}^2E_x}{\mathrm{d}z^2}+k^2E_x = 0, \tag{10·1}$$

$$\frac{\mathrm{d}^2E_y}{\mathrm{d}z^2}+k^2\cos^2\theta E_y = 0. \tag{10·2}$$

The solutions of these two equations will be used to connect the reflexion coefficients of region I referred to the height z_1, and the reflexion coefficients of region II referred to the height z_2. Evidently from (10·1) and (10·2) the factor $\exp[-\mathrm{i}k(z_2-z_1)]$ must be introduced for E_x waves between the two levels, and $\exp[-\mathrm{i}k(z_2-z_1)\cos\theta]$ for E_y waves.

In the free space below the ionosphere it is supposed that the incident wave in the (x, z) plane makes an angle θ with the vertical (z-axis) and is polarized with its electric vector in the (x, z) plane. Its field is therefore

$$E_x = \cos\theta\exp[-\mathrm{i}k(z\cos\theta+x\sin\theta)]\exp(\mathrm{i}kct),$$
$$E_z = -\sin\theta\exp[-\mathrm{i}k(z\cos\theta+x\sin\theta)]\exp(\mathrm{i}kct),$$
$$E_y = 0.$$

The wave which is finally returned from the ionosphere, after due account has been taken of the effects of the two regions, may be represented by

$$\left.\begin{aligned} E_x &= R_{xx}\exp(-2\mathrm{i}kz_1\cos\theta)\cos\theta\exp[-\mathrm{i}k(-z\cos\theta+x\sin\theta)]\exp(\mathrm{i}kct),\\ E_z &= R_{xx}\exp(-2\mathrm{i}kz_1\cos\theta)\sin\theta\exp[-\mathrm{i}k(-z\cos\theta+x\sin\theta)]\exp(\mathrm{i}kct),\\ E_y &= R_{xy}\exp(-2\mathrm{i}kz_1\cos\theta)\exp[-\mathrm{i}k(-z\cos\theta+x\sin\theta)]\exp(\mathrm{i}kct),\end{aligned}\right\} \tag{10·3}$$

where R_{xx} and R_{xy} are factors to which we shall give the names 'overall reflexion coefficient' and 'overall conversion coefficient' respectively. The factor $\exp(-2\mathrm{i}kz_1\cos\theta)$ is introduced throughout so that R_{xx} and R_{xy} are referred to the height z_1. By considering the successive reflexions between regions I and II, but neglecting waves which have been reflected one or more times by the ground, it can be seen that R_{xx} and R_{xy} are given by the following infinite geometrical progressions:

$$R_{xx} = r + t\rho_{xx}t' \exp[-2ik(z_2-z_1)] + t\rho_{xx}r'\rho_{xx}t' \exp[-4ik(z_2-z_1)] + \ldots, \quad (10\cdot4)$$

$$R_{xy} = t\rho_{xy} \exp[-ik(z_2-z_1)(1+\cos\theta)] + t\rho_{xx}r'\rho_{xy} \exp[-2ik(z_2-z_1)(1+\cos\theta)] + \ldots. \quad (10\cdot5)$$

If $|\rho_{xx}r'| \ll 1$, R_{xx} is represented with sufficient accuracy by the first two terms, R_{xy} by the first term of these series.

If the incident wave in the free space below the ionosphere is represented by

$$E_y = \exp[-ik(z\cos\theta + x\sin\theta)] \exp(ikct),$$
$$E_x = E_z = 0,$$

so that the electric vector is perpendicular to the plane of propagation, then it may be shown that the corresponding coefficients are given by

$$R_{yx} = \rho_{yx}t' \exp[-ik(z_2-z_1)(1+\cos\theta)], \quad (10\cdot6)$$

$$R_{yy} = \rho_{yy} \exp[-2ik(z_2-z_1)\cos\theta]. \quad (10\cdot7)$$

In applying formulae (10·4) to (10·7), it is sufficient to calculate $t\sqrt{\sec\theta} = t'\sqrt{\cos\theta}$ (see (6·7)) rather than t, t' separately, and $\rho_{xy}\sqrt{\cos\theta} = -\rho_{yx}\sqrt{\sec\theta}$ (see (9·3) and (9·7)) rather than ρ_{xx} and ρ_{yx}, since in formula (10·4) we can write

$$t\rho_{xx}t' = (t\sqrt{\sec\theta})^2 \rho_{xx},$$

and in (10·5) or (10·6)

$$t\rho_{xy} = -\rho_{yx}t' = (t\sqrt{\sec\theta})(\rho_{xy}\sqrt{\cos\theta})$$

We note also that $R_{xy} = -R_{yx}$.

The coefficients R_{xx}, R_{xy}, R_{yx}, R_{yy} thus represent the overall reflexion and conversion coefficients of the whole ionosphere, and include the effects of both regions I and II. The moduli of the R's represent the amplitudes of the downcoming waves and their arguments represent the phases. These phases are related to the apparent height of reflexion, and in the next section we shall consider what can be deduced from them about this height.

11. Apparent height

For the measurement of the apparent height of reflexion on long and very long waves, the simplest method is that of the 'Hollingworth' interference pattern (Hollingworth 1926). In this method, measurements are made on the interference pattern formed at the ground by the combination of the ground and downcoming waves. In its most usual form the amplitude $|E_z|$ of the resultant vertical electric field is measured as a function of distance (x), and the product $|E_z|\,x$ is plotted against x so as to remove the inverse distance factor appropriate to the spreading of waves in free space. The curve shows characteristic maxima and minima, and it is usual to describe it in terms of a fictitious model ionosphere in which reflexion takes place at a sharp boundary, at a definite height, and possibly accompanied by a phase change.

Suppose the upgoing wave to be polarized with its electric field in the vertical plane, and observations to be made on this same component of field in the downcoming wave. The appropriate reflexion coefficient is the one which we have called R_{xx}, and we shall write $\phi = \arg R_{xx}$. If ϕ is independent of the angle θ, we may describe the experimental results in terms of a fictitious horizonal reflector. If, however, ϕ depends on θ, we must explain the results in terms either of a reflector whose height, or whose phase-shift, depends on θ.

Equation (10·3) gives the relative phase of the reflected vertical component of the field; for brevity denote $\arg E_z$ by χ. Then

$$\chi = \phi - 2kz_1 \cos\theta - k(-z\cos\theta + x\sin\theta),$$

omitting the kct. Taking the transmitter at the origin, and measuring the phase of the reflected wave at the ground where $z = 0$, then

$$\chi = \chi(x, \theta) = \phi - 2kz_1 \cos\theta - kx\sin\theta. \tag{11·1}$$

As shown by Hartree (1931), the emitted wave may be resolved into trains of plane waves travelling in different directions. Each plane wave gives rise to a plane reflected wave which will reach the receiver, but only those for which the phase is stationary with respect to the direction of the wave normal will contribute appreciably to the resultant field there. The value of θ which will contribute most to the effect at the receiver is the root of

$$\partial\chi/\partial\theta = 0 \tag{11·2}$$

or

$$\partial\phi/\partial\theta + 2kz_1 \sin\theta - kx\cos\theta = 0. \tag{11·3}$$

Denote this value of θ by θ_a. The apparent height z_a is defined for the purpose of this paper as the height which the ray would have reached if it had been reflected sharply at an angle of incidence θ_a, i.e.

$$z_a = \tfrac{1}{2}x\cot\theta_a. \tag{11·4}$$

Solving (11·3) for x, and substituting in (11·4),

$$z_a - z_1 = \frac{\operatorname{cosec}\theta_a}{2k}\left[\frac{\partial\phi}{\partial\theta}\right]_{\theta=\theta_a} = -\frac{1}{2k}\left[\frac{\partial\phi}{\partial(\cos\theta)}\right]_{\theta=\theta_a}$$

$$= -\frac{1}{2k}\frac{\pi}{180}\left[\frac{\partial\phi^\circ}{\partial(\cos\theta)}\right]_{\theta=\theta_a}, \tag{11·5}$$

when ϕ° is expressed in degrees. Note that $z_a = z_1$ only if

$$[\partial\phi/\partial\theta]_{\theta=\theta_a} = 0.$$

Experimentally, the angle θ_a, and hence from (11·4) z_a, might be measured as follows. The total rate of change of χ in (11·1) with respect to x is

$$\frac{d\chi}{dx} = \frac{\partial\chi}{\partial x} + \frac{\partial\chi}{\partial\theta}\frac{\partial\theta}{\partial x}.$$

But from (11·2) $[\partial\chi/\partial\theta]_{\theta=\theta_a} = 0$; hence

$$\frac{d\chi}{dx} = \frac{\partial\chi}{\partial x} = -k\sin\theta_a \tag{11·6}$$

from (11·1). Since χ could be found experimentally as a function of x, the angle of incidence θ_a can be derived from (11·6). It is interesting to note that $d\chi/dx$ could be measured with a spaced loop direction finder modified to receive the sky wave only, and also

θ_a could be measured by comparing the field measured by a vertical loop aerial and a vertical aerial as in 'sense' measurements in direction finding; but both methods are subject to experimental difficulties.

12. The parameters describing the ionosphere

In § 13 we shall present, in graphical form, the results of numerical calculations for a model which approximates to that of the lowest part of the ionosphere. In this section we discuss the magnitudes of the parameters used to describe this part of the ionosphere.

We assume that the electron density, and hence X, is distributed as in the 'tail' of a Chapman (1931) region, and hence we write, using an approximation given by Budden, Ratcliffe & Wilkes (1939), which holds in the lowest part of such a region,

$$X = X_0 \exp\{-\tfrac{1}{2}\exp[-(z-z_0)/H]\}, \tag{12·1}$$

where H is the scale height of the atmosphere, and z_0 is the level at which the maximum ionization occurs. We assume that the collision frequency, and hence Z, is given by

$$Z_0 \exp(-z/H).$$

Region I

We now use (12·1) to find a value for the quantity α (see (4·8)) in terms of the scale height H in region I. This region is situated near the level z_1, where $|X/(1-\mathrm{i}Z)| = 1$. If X and Z vary in the way described above, the rate of variation of X is so much greater than that of Z that we can assume Z to be constant over region I. We define ϵ as the value of

$$\arg[X/(1-\mathrm{i}Z)]$$

at the height $z = z_1$. For heights near z_1, for which $(z-z_1)/H$ is small, we can rewrite (12·1) in the following form:

$$\begin{aligned} X &= X_0 \exp\left\{-\tfrac{1}{2}\exp\left[-\frac{(z-z_1)}{H}-\frac{(z_1-z_0)}{H}\right]\right\} \\ &\doteqdot X_0 \exp\left\{-\frac{1}{2}\left(1-\frac{z-z_1}{H}\right)\exp\left[-\frac{(z_1-z_0)}{H}\right]\right\} \\ &= X_0 \exp\left\{-\tfrac{1}{2}\exp\left[-\frac{(z_1-z_0)}{H}\right]\right\}\exp\left\{\frac{(z-z_1)}{2H}\exp\left[-\frac{(z_1-z_0)}{H}\right]\right\}. \end{aligned} \tag{12·2}$$

When $z = z_1$, $$X = |1-\mathrm{i}Z|,$$

and (12·2) becomes $$X_0 \exp\left\{-\tfrac{1}{2}\exp\left[-\frac{(z_1-z_0)}{H}\right]\right\} = |1-\mathrm{i}Z|.$$

Rearrangement gives

$$\frac{1}{2H}\exp\left[-\frac{(z_1-z_0)}{H}\right] = \frac{1}{H}\log\frac{X_0}{|1-\mathrm{i}Z|} = \alpha,$$

this being the definition of α.

Substitution in (12·2) gives

$$X = |1-\mathrm{i}Z| \exp[\alpha(z-z_1)],$$

or

$$\frac{X}{1-\mathrm{i}Z} = \exp(\mathrm{i}\epsilon)\exp[\alpha(z-z_1)]. \tag{12·3}$$

Table 1 shows the numerical magnitudes of the quantities required to describe region I. The magnitude for H is taken from recent experimental work (Best, Havens & La Gow 1947). The value of $X_0 = p_0^2/p^2$ is derived from the assumption that the penetration frequency $p_0/2\pi$ of the E layer is 1·27 Mc/s. $(z_0 - z_1)$ is calculated from (12·1), by assuming that $X = 1$ at $z = z_1$. It should be pointed out that, for simplifying the numerical computations, z_1 and α have been calculated for $Z = 0$, and these values have been used even when the collision frequency is appreciable, as it was found that for the values of Z used it was the ϵ terms and not the otherwise slight variation in α that affected the calculated coefficients. The value $(5 \times 10^5\,\mathrm{s}^{-1})$ taken for ν is such as to make $\epsilon = \frac{1}{4}\pi$ for a wave frequency of 80 kc/s. If ν had been taken much greater, then the approximations needed in § 3 for the separation into two regions would not have been valid up to 80 kc/s. It is probable that these assumptions about ν will only apply to winter night conditions.

TABLE 1

f	16 kc/s	80 kc/s
λ	18·75 km	3·75 km
$k = 2\pi/\lambda$	0·335 km^{-1}	1·675 km^{-1}
H	6 km	6 km
X_0	10^5	4×10^3
$z_0 - z_1$	18·82 km	16·85 km
α	1·92 km^{-1}	1·38 km^{-1}
$h = k/\alpha$	0·174	1·21
ϵ	$\frac{1}{2}\pi$ (approx.)	$\frac{1}{4}\pi$

Region II

We define z_3 by the relation $X(z_3) = |Y|$, so that from (12·1),

$$z_0 - z_3 = H \log_e (2 \log_e X_0 / |Y|),$$

and if $(z - z_3)/H$ is small, this gives us, as before,

$$\frac{X}{|Y|} = \exp\left[\frac{(z-z_3)}{H} \log_e \frac{X_0}{|Y|}\right].$$

We now introduce γ, as defined in § 7,

$$\gamma = \frac{1}{H} \log_e \frac{X_0}{|Y|}.$$

In § 7 it was convenient to work in terms of $(z - z_2)$ rather than $(z - z_3)$, where

$$z_2 = z_3 + \frac{2}{\gamma} \log_e \frac{2\gamma}{k}.$$

Then
$$\frac{X}{|Y|} = \frac{4\gamma^2}{k^2} \exp[\gamma(z - z_2)]. \tag{12·4}$$

Table 2 shows the numerical magnitudes of the quantities needed to describe region II. Here $z_0 - z_3$ is calculated from (12·1) by writing $X = |Y|$ at z_3, and $z_3 - z_1$ follows by comparison with table 1. $z_3 - z_1$ is the distance between the levels which define regions II and I; but although phases are referred to the level z_1 for region I, they are referred to z_2 and not z_3 for region II.

TABLE 2

f	16 kc/s	80 kc/s
X_0	10^5	4×10^3
$\lvert Y\rvert$	80	16
$X_0/\lvert Y\rvert$	1250	250
z_0-z_3	15·94 km	14·41 km
z_3-z_1	2·88 km	2·44 km
γ	1·19 km^{-1}	0·92 km^{-1}
$a=k/2\gamma$	0·140	0·91
z_2-z_3	3·31 km	0·205 km
z_2-z_1	6·19 km	2·64 km
$k(z_2-z_1)$	2·07 radians (=118·6°)	4·42 radians (=253·3°)

The assumed distribution of X with height, and the heights z_1, z_2 and z_3 are shown for frequencies of 16 and 80 kc/s in figures 23 and 24 respectively. The approximations appropriate to regions I and II respectively hold inside the areas shaded in the figures. The tangents at the heights z_1 and z_3 show the approximations

$$X = \exp[\alpha(z-z_1)],$$

$$X = \lvert Y\rvert \exp[\gamma(z-z_3)],$$

used in this analysis. In both figures, X changes by a factor of order 5 before the approximations are seriously in error.

13. GRAPHICAL REPRESENTATION OF THE RESULTS

The model ionosphere discussed in the previous section, having the parameters listed in tables 1 and 2, has been used in a series of numerical calculations made for the purpose of illustrating the results of the theory. The calculations were made as follows. The reflexion coefficients ($\rho_{xx}, \rho_{xy}, \rho_{yx}, \rho_{yy}, r, r'$) and transmission coefficients (t, t') for the two regions were calculated from the expressions given in §§ 6 and 9. In order to calculate the interference effects produced by the combination of the waves reflected from the two regions, use has been made of (10·1) and (10·2), which show that for the vertically polarized component the effective path difference between the reference levels z_1 and z_2 is (z_2-z_1), whereas for the horizontally polarized component it is $(z_2-z_1)\cos\theta$.

For the calculation of the phases, numerical values are required for functions of the forms $\arg(\mathrm{i}y)!$ and $\arg(\mathrm{i}y-\frac{1}{2})!$, where y is real. Stieltjes (1886) has given values of $\arg(\mathrm{i}y)!$ for $y = 0(0{\cdot}2)\,4$. The values required lie in this range, and can be obtained graphically or by interpolation from his values. To calculate $\arg(\mathrm{i}y-\frac{1}{2})!$, a direct application of Legendre's duplication formula (Copson 1935, § 9·23) gives

$$\arg(\mathrm{i}y-\tfrac{1}{2})! = \arg(2\mathrm{i}y)! - \arg(\mathrm{i}y)! - y\log_e 4,$$

the range of this being $y = 0(0{\cdot}2)\,2$ calculated from Stieltjes's values of $\arg(\mathrm{i}y)!$. For larger values of y, $\arg(\mathrm{i}y-\frac{1}{2})!$ can be calculated from its asymptotic form. A short table is given in appendix 2 of $\arg(\mathrm{i}y)!$ and $\arg(\mathrm{i}y-\frac{1}{2})!$ for $y = 0(0{\cdot}2)\,4$.

The results are presented graphically in figures 2 to 22, pp. 495 to 498, for the two frequencies 16 and 80 kc/s. The graphs show the way in which various quantities vary with the cosine of the angle of incidence of the wave, and ordinates corresponding to angles of 0, 30, 60 and 90° are indicated. The horizontal scale is $(1-\cos\theta)$ on all graphs.*

Region I exerts no influence on waves polarized with their electric vectors perpendicular to the plane of incidence. Waves polarized with their electric vectors parallel to the plane of incidence correspond to reflexion coefficients r, r'; and transmission coefficients t, t'; and they remain polarized with their electric vector in the plane of incidence. The transmission coefficients for upgoing (t) and downgoing (t') waves are related by the expression

$$t\sqrt{\sec\theta} = t'\sqrt{\cos\theta}.$$

In figures 2, 3 and 4, which are drawn for the frequency of 16 kc/s, curves are shown for the two cases where

(*a*) $\epsilon = 0$, corresponding to $Z \ll 1$ or $\nu \ll p$,

(*b*) $\epsilon = \frac{1}{2}\pi$, corresponding to $|Y| \gg Z \gg 1$, or $\nu \gg p$;

and it is seen that the magnitudes are only slightly different for these two values of ϵ. This occurs because the thickness of region I is only a fraction of the wave-length when the frequency is 16 kc/s, so that the coefficients depend principally on the magnitude of the discontinuity in the effective refractive index, and not on the detailed nature of the transition.

The phases of the two transmission coefficients and of the upwards reflexion coefficient are very small for all angles of incidence, while the phase of the downward reflexion coefficient is very close to π.

Figures 12, 13 and 14 are drawn for a frequency of 80 kc/s, both for $\epsilon = 0$ and for $\epsilon = \frac{1}{4}\pi$. The reason for the choice $\epsilon = \frac{1}{4}\pi$ is that the collision frequency has been taken to be the same for both frequencies; this was discussed in connexion with table 1, § 12. It would be impossible to use $\epsilon = \frac{1}{2}\pi$ for 80 kc/s, since the relation $\epsilon = \tan^{-1} Z$ would require Z to be large, in conflict with the requirement $|Y| \gg Z$ ($Y = 16$ for 80 kc/s) necessary for the splitting into two regions. For this frequency, and for both values of ϵ, the downward reflexion coefficient is negligibly small. It will be noted that there is a marked difference between the reflexion coefficients for $\epsilon = 0$ and $\epsilon = \frac{1}{4}\pi$; this arises from the fact that the thickness of the region is comparable with the wave-length at 80 kc/s.

Waves of both polarizations are reflected from region II, and both have their polarizations altered by reflexion, so that both reflexion and conversion coefficients are significant. The two conversion coefficients are related by the equations

$$|\rho_{xy}|\sqrt{\cos\theta} = |\rho_{yx}|\sqrt{\sec\theta},$$

and

$$\arg\rho_{xy} = \mp\pi + \arg\rho_{yx}.$$

The curves of figures 5 and 6 show how the magnitudes and phases of the coefficients vary with angle for the waves of frequency 16 kc/s, and figures 15 and 16 relate to a frequency of 80 kc/s. Since Z does not enter into the equations of propagation for this region (see (7·1) and (7·2)), the curves are valid for all values of ϵ consistent with the condition $|Y| \gg Z$.

* The advantage of using $(1-\cos\theta)$ rather than θ for the horizontal scale, is that most of the graphs approximate more closely to straight lines in this form, and calculation of the apparent height by graphical differentiation is simplified.

The next series of curves relates to the resultant downcoming wave as it arrives at the ground after modification by both the regions. The calculations leading to these curves were made from the formulae given in § 10, using the coefficients shown in figures 2 to 6 and 12 to 16 for the two regions separately. Figures 7 and 8 show the magnitudes of the reflexion and conversion coefficients R_{xx} and R_{xy} for 16 kc/s, when $\epsilon = 0$ and $\epsilon = \frac{1}{2}\pi$ respectively. R_{yx} can be obtained from the relation

$$|R_{xy}| = |R_{yx}|,$$

and since region I has no influence on a wave with its electric vector along the y-direction, we have

$$|R_{yy}| = |\rho_{yy}|,$$

and its magnitude can be obtained from figure 5. Figures 9 and 10 show the phases of these coefficients. In figure 11 the 'apparent' reflexion heights, as deduced by applying (11·5) to the curves of figures 9 and 10, are shown for the different reflected waves.

The following features are noticeable in the curves of figures 7, 8, 9, 10 and 11, drawn for waves of frequency 16 kc/s.

(*a*) The reflexion and conversion coefficients vary monotonically with angle of incidence, and do not show a minimum for some intermediate angle.

(*b*) There is no radical difference between the case when $\epsilon = 0$ ($\nu = 0$) and $\epsilon = \frac{1}{2}\pi$ (ν large).

(*c*) The apparent height of reflexion is remarkably independent of the angle of incidence.

Figures 17, 18, 19, 20, 21 and 22 refer similarly to waves of 80 kc/s. The following points are noticeable.

(*a*) The reflexion coefficient $|R_{xx}|$ (figures 17, 18) passes through a minimum at some intermediate angle of incidence. Though reminiscent of the behaviour of optical waves near the Brewster angle, there is no connexion between the two phenomena. The Brewster effect is purely a reflexion effect at a plane surface, but in figures 17 and 18 the minimum arises through the interference between the waves reflected from the two regions. The monotonic variation of $|R_{xx}|$ with angle of incidence for 16 kc/s in figures 7 and 8 is strictly speaking an interference effect, and variations in the parameters describing the ionosphere will alter these interference effects considerably. This minimum is accompanied by a rapid change of phase (figure 20), which in turn leads to quite important changes of apparent height (figure 21) as the angle of incidence is varied.

(*b*) The magnitude and phase of R_{xx} depend quite considerably on the value of ϵ, but the minimum effect occurs both for $\epsilon = 0$ and $\epsilon = \frac{1}{4}\pi$.

(*c*) The magnitude of $|R_{xy}|$ differs little in the two cases, and the phases of the other coefficients are independent of the value of ϵ.

(*d*) For the other polarizations, the apparent height does not change much with angle of incidence.

We conclude with a brief comparison with experimental results; for a complete survey of these results the reader is referred to Bracewell, Budden, Ratcliffe, Straker & Weekes (1951). Comparison is made for night-time conditions only. A qualitative comparison only can be given; a quantitative comparison would depend more critically on the values of the parameters chosen in § 12.

Experimentally, the reflexion coefficient $|R_{xx}|$ is of the order 0·5 on 16 kc/s, and of the order 0·4 on 80 kc/s measured at a distance of 90 km from the transmitter. For larger distances, $|R_{xx}|$ increases slightly with increasing distance for 16 kc/s, and $|R_{xx}|$ for 80 kc/s increases to about 0·5 at approximately 1000 km from the transmitter. The theory predicts the gradual increase of $|R_{xx}|$ for 16 kc/s (see figure 8), and likewise figure 18 demonstrates the increase of $|R_{xx}|$ for 80 kc/s with large angles of incidence, once the dip is passed. We can make no statement concerning this dip (caused by interference between the two regions); results of observations on vertical and oblique incidence propagation during the same night are not available.

Experimentally, it is found that at vertical incidence, $|R_{xy}|$ decreases slightly with increase of frequency under night-time conditions. The theory does not predict this decrease; for at vertical incidence $|R_{xy}| = |\rho_{xy}|$ tends asymptotically with frequency to the value of 0·5 (see (9·1)). This lack of agreement can be accounted for either by assuming, as suggested in § 1, that there is an absorbing region at a level below those considered in this paper, or by the fact that there is a slight effect in region II caused by the collision frequency; but the approximations needed for the two regions to separate demand that this collision frequency should be neglected.* For vertical incidence only, Stanley (1950) has given the complete solution for the case of an exponential variation of electron density with height, and his formulae show there is no asymptotic approach to the value 0·5 provided the collision frequency is sufficient.

Experimentally, the apparent height corresponding to R_{xx} is known within 5 km to be the same for 16 and 80 kc/s as measured at a distance of 90 km. For angles of incidence up to 65° on 16 kc/s, the apparent height is constant, but for angles greater than 65° results cannot be obtained, due to the multiple reflexion of waves and to fading. For frequencies near 80 kc/s the results analyzed so far for night-time conditions suggest that the apparent height falls slightly with increase of angle of incidence for angles greater than about 45°.† Theoretically, figure 11 suggests this constancy of height for 16 kc/s, and figure 20 shows a fall in the apparent height for angles of incidence well away from the vertical. No results are available to verify the rapid change in height very near the vertical.

It is interesting to notice that the only one of the four overall coefficients that is independent of the lower region is ρ_{yy} (see (10·7)); information about the character of region II, free from the effects of region I, could be obtained from this coefficient; but it is just that coefficient which cannot be measured experimentally with existing transmitters.

* It should be remembered also that we have assummed that the earth's magnetic field is vertical; i.e. the theory developed in this paper only refers to high geomagnetic latitudes.

† The authors are indebted to Dr K. Weekes for a private communication concerning his analysis of 80 kc/s.

Region I, 16 kc/s

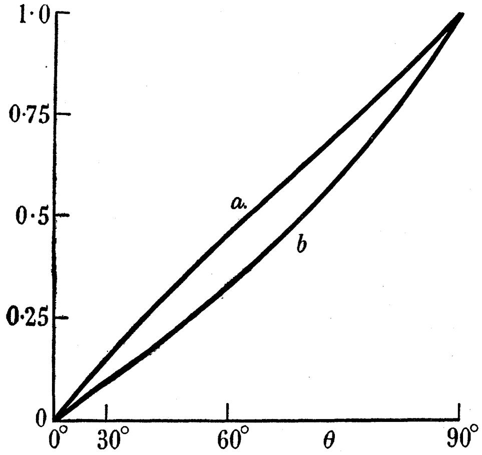


FIGURE 2

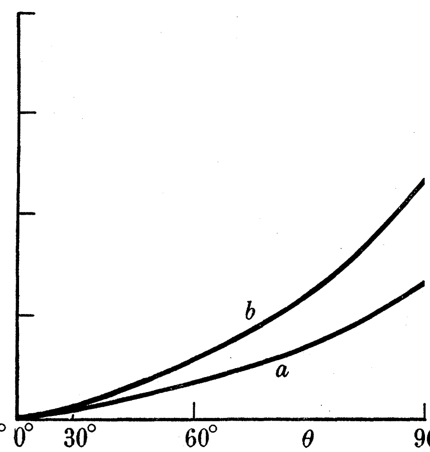


FIGURE 3

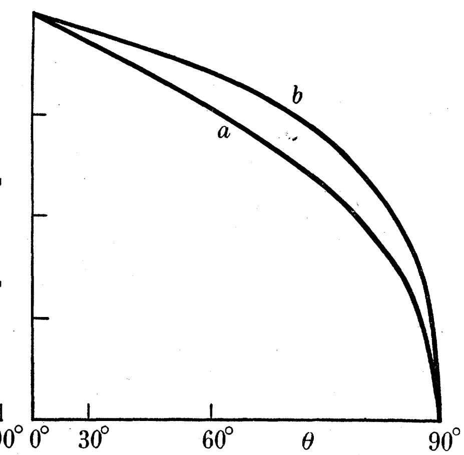


FIGURE 4

FIGURE 2. The modulus of the upward reflexion coefficient. (To within 1°, the phase of r referred to height z_1 is zero.) (a) $\epsilon=0$; (b) $\epsilon=\frac{1}{2}\pi$.

FIGURE 3. The modulus of the downward reflexion coefficient. (To within 1°, the phase of r' is π, when referred to height z_1.) (a) $\epsilon=0$; (b) $\epsilon=\frac{1}{2}\pi$.

FIGURE 4. The modulus of the transmission coefficients, $t\sqrt{\sec\theta}=t'\sqrt{\cos\theta}$. (To within 1°, the phase of t and t' is zero when referred to height z_1.) (a) $\epsilon=0$; (b) $\epsilon=\frac{1}{2}\pi$.

Region II, 16 kc/s

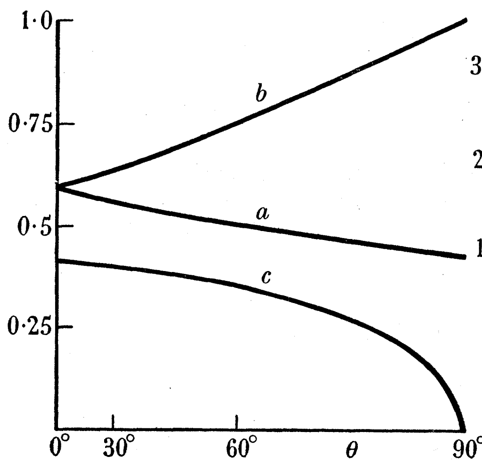


FIGURE 5

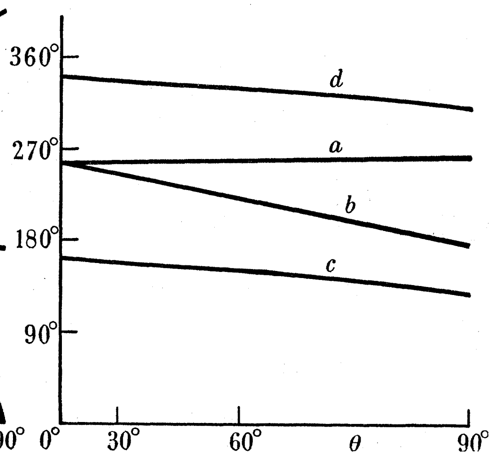


FIGURE 6

FIGURE 5. The moduli of the reflexion and conversion coefficients.

(a) $|\rho_{xx}|$; (b) $|\rho_{yy}|$; (c) $|\rho_{xy}|\sqrt{\cos\theta}=|\rho_{yx}|\sqrt{\sec\theta}$.

FIGURE 6. The phases of the reflexion and conversion coefficients referred to height z_2.

(a) $\arg\rho_{xx}$; (b) $\arg\rho_{yy}$; (c) $\arg\rho_{xy}$; (d) $\arg\rho_{yx}$.

Overall reflexion, 16 kc/s

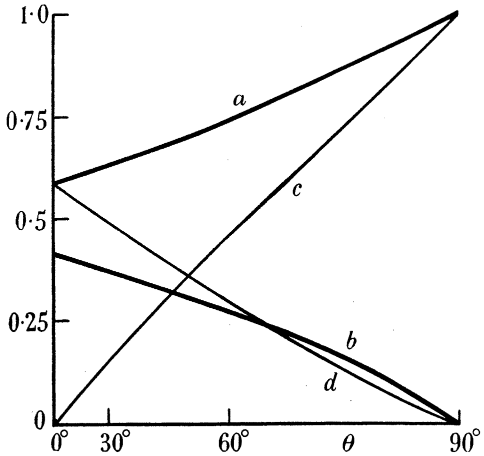


FIGURE 7

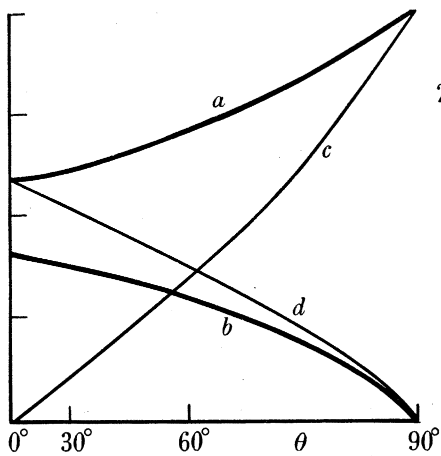


FIGURE 8

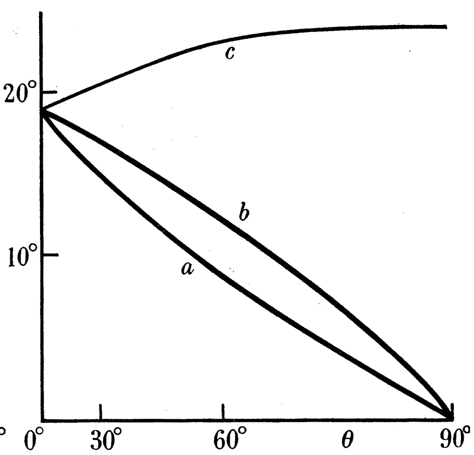


FIGURE 9

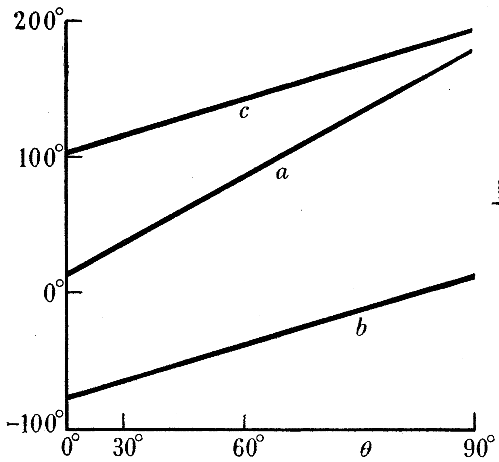


FIGURE 10

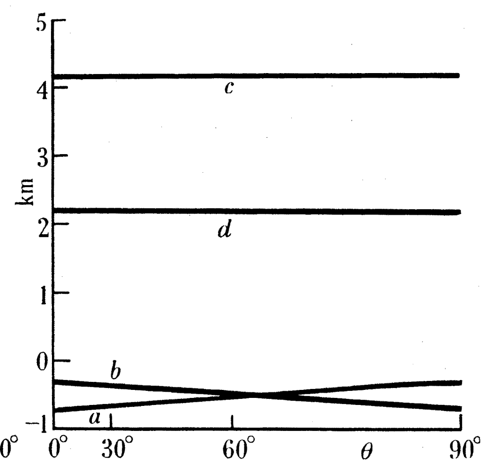


FIGURE 11

FIGURE 7. The moduli of the overall reflexion and conversion coefficients, showing also the moduli of the two separate parts of R_{xx} (for the case $\epsilon=0$). (*a*) $|R_{xx}|$; (*b*) $|R_{xy}|=|R_{yx}|$; (*c*) $|r|$; (*d*) $|t\rho_{xx}t'|$. Note $|R_{yy}|=|\rho_{yy}|$.

FIGURE 8. The moduli of the overall reflexion and conversion coefficients, showing also the moduli of the two separate parts of R_{xx} (for the case $\epsilon=\frac{1}{2}\pi$). (*a*) $|R_{xx}|$; (*b*) $|R_{xy}|=|R_{yx}|$; (*c*) $|r|$; (*d*) $|t\rho_{xx}t'|$. Note $|R_{yy}|=|\rho_{yy}|$.

FIGURE 9. The phase of the overall reflexion coefficient R_{xx} referred to height z_1. (*a*) arg R_{xx}, $\epsilon=0$; (*b*) arg R_{xx}, $\epsilon=\frac{1}{2}\pi$; (*c*) arg $\{t\rho_{xx}t' \exp[-2ik(z_2-z_1)]\}$.

FIGURE 10. The phase of the overall coefficients referred to height z_1, for $\epsilon=0$ and $\epsilon=\frac{1}{2}\pi$. (*a*) arg R_{yy}; (*b*) arg R_{xy}; (*c*) arg R_{yx}.

FIGURE 11. The apparent height corresponding to the overall coefficients, measured with respect to the height z_1. (*a*) $(z_{ap})_{xx}$, $\epsilon=0$; (*b*) $(z_{ap})_{xx}$, $\epsilon=\frac{1}{2}\pi$; (*c*) $(z_{ap})_{yy}$; (*d*) $(z_{ap})_{xy}=(z_{ap})_{yx}$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Region I, 80 kc/s

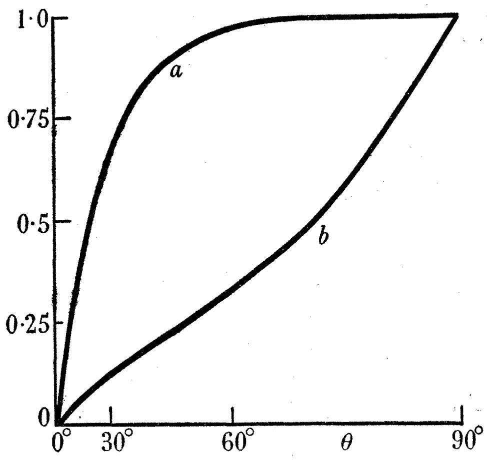


Figure 12

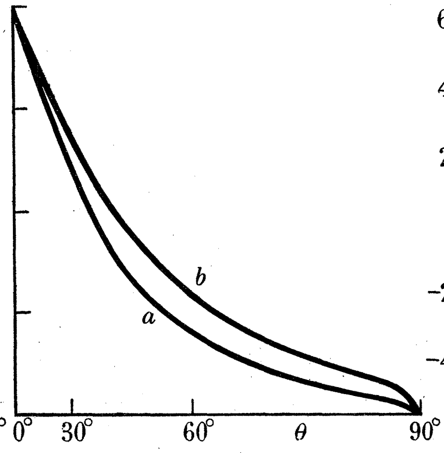


Figure 13

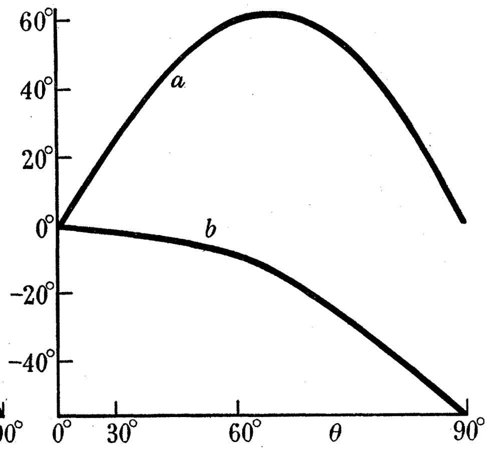


Figure 14

Figure 12. The modulus of the upward reflexion coefficient. (The modulus of the downward reflexion coefficient r' is negligible both for $\epsilon=0$ and $\epsilon=\frac{1}{4}\pi$.) (*a*) $\epsilon=0$; (*b*) $\epsilon=\frac{1}{4}\pi$.

Figure 13. The modulus of the transmission coefficients, $t\sqrt{\sec\theta}=t'\sqrt{\cos\theta}$. (*a*) $\epsilon=0$; (*b*) $\epsilon=\frac{1}{4}\pi$.

Figure 14. The phases of the reflexion and conversion coefficients, both the $\epsilon=0$ and $\epsilon=\frac{1}{4}\pi$, referred to height z_1. (*a*) arg r; (*b*) arg $t=$ arg t'.

Region II, 80 kc/s

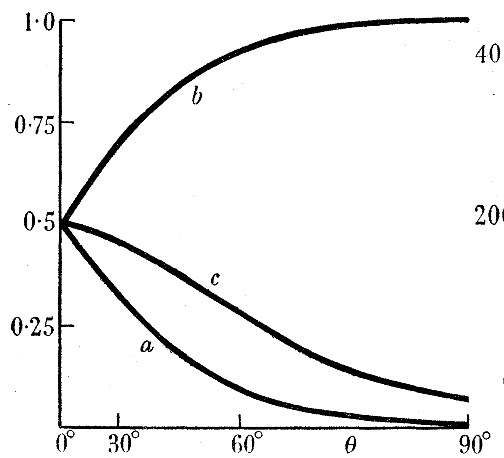


Figure 15

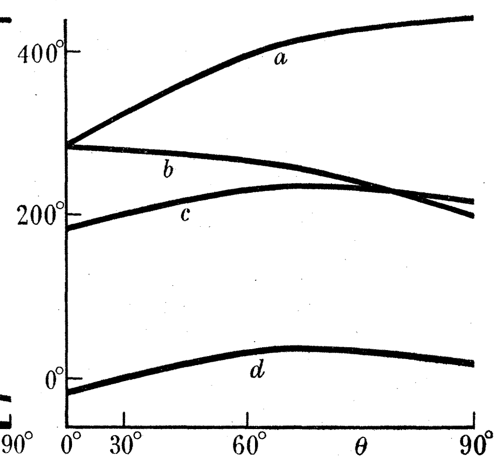


Figure 16

Figure 15. The moduli of the reflexion and conversion coefficients. (*a*) $|\rho_{xx}|$; (*b*) $|\rho_{yy}|$; (*c*) $|\rho_{xy}|\sqrt{\cos\theta}=|\rho_{yx}|\sqrt{\sec\theta}$.

Figure 16. The phases of the reflexion and conversion coefficients referred to height z_2. (*a*) arg ρ_{xx}; (*b*) arg ρ_{yy}; (*c*) arg ρ_{xy}; (*d*) arg ρ_{yx}.

Overall reflexion, 80 kc/s

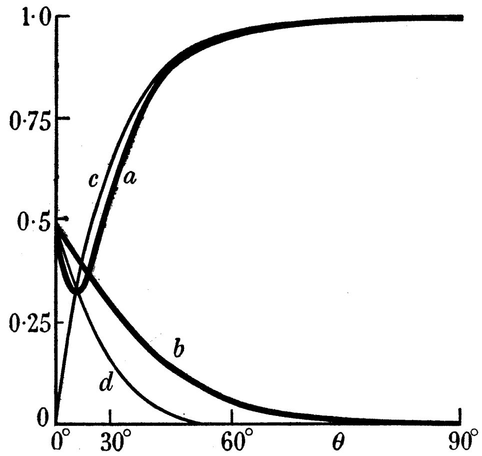


FIGURE 17

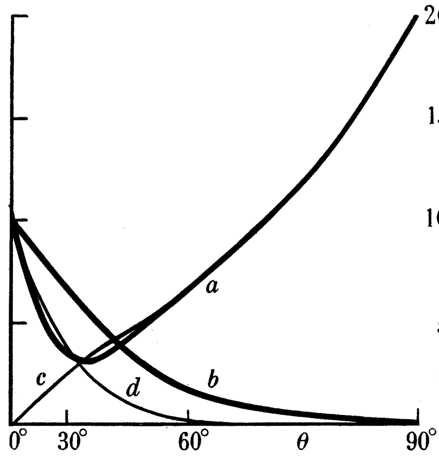


FIGURE 18

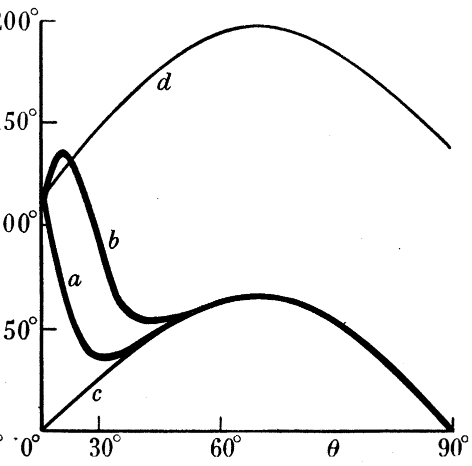


FIGURE 19

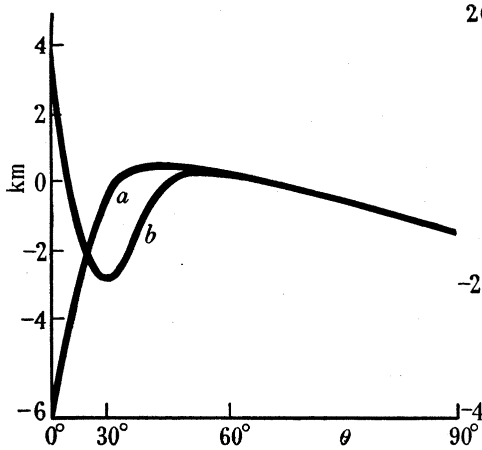


FIGURE 20

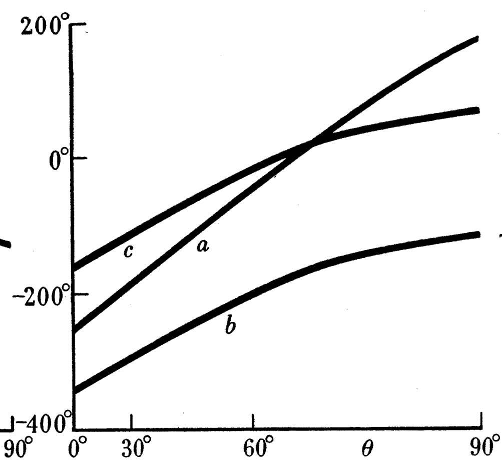


FIGURE 21

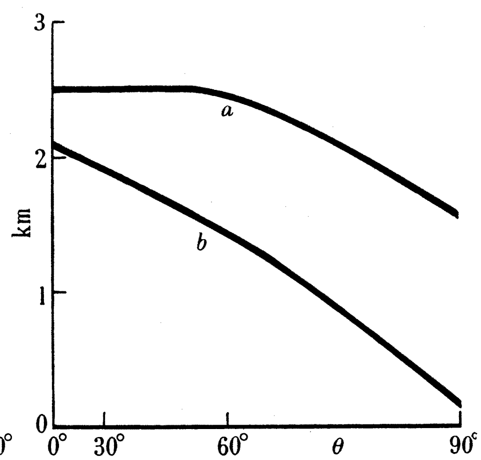


FIGURE 22

FIGURE 17. The moduli of the overall reflexion and conversion coefficients, showing the moduli of the two separate parts of R_{xx} (for the case $\epsilon = 0$). (*a*) $|R_{xx}|$; (*b*) $|R_{xy}| = |R_{yx}|$; (*c*) $|r|$; (*d*) $|t\rho_{xx}t'|$. Note $|R_{yy}| = |\rho_{yy}|$.

FIGURE 18. The moduli of the overall reflexion and conversion coefficients, showing the moduli of the two separate parts of R_{xx} (for the case $\epsilon = \frac{1}{4}\pi$). (*a*) $|R_{xx}|$; (*b*) $|R_{xy}| = |R_{yx}|$; (*c*) $|r|$; (*d*) $|t\rho_{xx}t'|$.

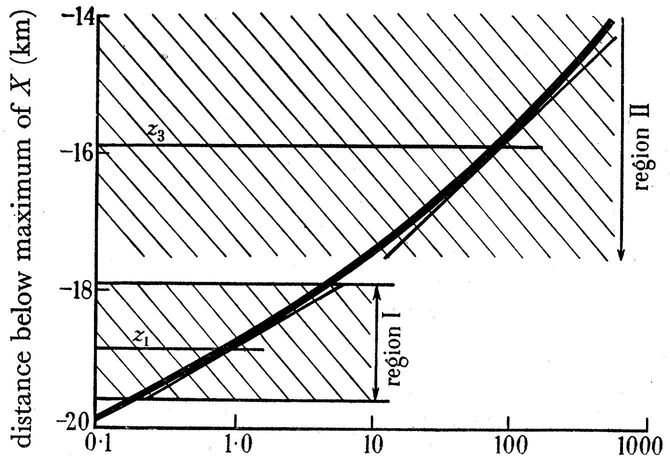
FIGURE 19. The phase of the overall reflexion coefficient R_{xx} referred to height z_1. (*a*) arg R_{xx}, $\epsilon = 0$; (*b*) arg R_{xx}, $\epsilon = \frac{1}{4}\pi$; (*c*) arg r; (*d*) arg $\{t\rho_{xx}t' \exp[-2ki(z_2 - z_1)]\}$.

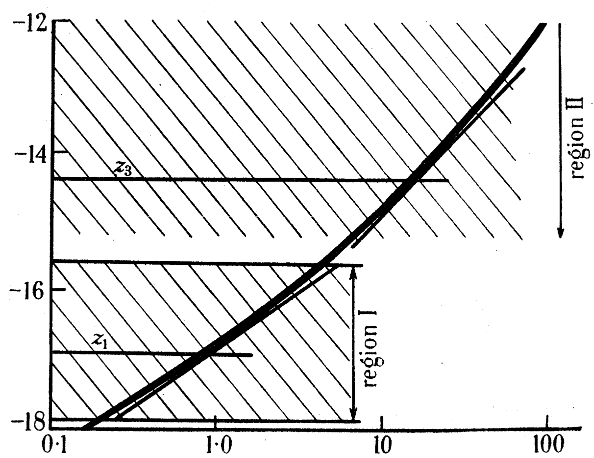
FIGURE 20. The apparent height corresponding to the overall coefficient R_{xx}, measured with respect to the height z_1. (*a*) $\epsilon = 0$; (*b*) $\epsilon = \frac{1}{4}\pi$.

FIGURE 21. The phases of the overall coefficients referred to height z_1, for $\epsilon = 0$ and $\epsilon = \frac{1}{4}\pi$. (*a*) arg R_{yy}; (*b*) arg R_{xy}; (*c*) arg R_{yx}.

FIGURE 22. The apparent height corresponding to the overall coefficients, measured with respect to the height z_1. (*a*) $(z_{ap})_{yy}$; (*b*) $(z_{ap})_{xy} = (z_{ap})_{yx}$.

Distribution of electron density





X (on logarithmic scale)

FIGURE 23 FIGURE 24

FIGURE 23. The variation of X at the bottom of a Chapman region, showing the exponential approximations (the tangents to the curve) valid in the two regions; for 16 kc/s.

FIGURE 24. The variation of X at the bottom of a Chapman region, showing the exponential approximations (the tangents to the curve) valid in the two regions; for 80 kc/s.

APPENDIX 1. ASYMPTOTIC FORMULA FOR $I_0(w)$

In this appendix, we obtain the asymptotic formula for the function $I_0(w)$ defined by the integral (8·3). Define

$$f(s) = \log\left[w^{-s}\prod_{j=1}^{4}(s+p_j-1)!\right]; \tag{A. 1}$$

then the integral to be evaluated is

$$I_0(w) = \frac{1}{2\pi \mathrm{i}}\int_{C_0}\exp\left[f(s)\right]\mathrm{d}s. \tag{A. 2}$$

Since the path C_0 (unlike the paths C_1, C_2, C_3, C_4) need not be drawn near the negative real axis, Stirling's formula may be used to evaluate the factorial functions. Stirling's formula states that if

$$|U| \gg 1 \quad \text{and} \quad |\arg U| < \pi,$$

then

$$U! \sim \sqrt{(2\pi)}\,(U+\tfrac{1}{2})^{(U+\frac{1}{2})}\exp\left[-(U+\tfrac{1}{2})\right].$$

Therefore

$$\log U! \sim \tfrac{1}{2}\log 2\pi + (U+\tfrac{1}{2})\left[\log(U+\tfrac{1}{2})-1\right], \tag{A. 3}$$

$$\frac{\mathrm{d}}{\mathrm{d}U}\log U! \sim \log(U+\tfrac{1}{2}),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}U^2}\log U! \sim 1/(U+\tfrac{1}{2}). \tag{A. 4}$$

Using (A. 3), $f(s)$ defined in (A. 1) may be expressed as

$$f(s) \sim 2\log 2\pi + \Sigma_j(s+p_j-\tfrac{1}{2})\left[\log(s+p_j-\tfrac{1}{2})-1\right] - s\log w. \tag{A. 5}$$

The integral (A. 2) will now be evaluated by the method of steepest descents. It is necessary to know the saddle points of $f(s)$, which are given by the roots of the equation

$$f'(s) = 0,$$

and the values of $f(s)$ and $f''(s)$ at these points. Differentiating (A. 5),

$$f'(s) \sim \Sigma_j \log(s+p_j-\tfrac{1}{2}) - \log w$$
$$= \log[\Pi_j(s+p_j-\tfrac{1}{2})/w], \tag{A. 6}$$

and
$$f''(s) \sim \Sigma_j 1/(s+p_j-\tfrac{1}{2}). \tag{A. 7}$$

The saddle points are therefore included amongst the roots of the equation

$$\Pi_j(s+p_j-\tfrac{1}{2}) = w.$$

If $|w| \gg \text{Max}[|p_1|, |p_2|, |p_3|, |p_4|, 1]$, these roots are given by

$$s = s_m = w^{\frac{1}{4}} \exp(\tfrac{1}{2}m\pi i) + \tfrac{1}{2} - \tfrac{1}{4}\Sigma_j p_j + O(w^{-\frac{1}{4}}).$$

On substituting this result into (A. 6), it can be seen that only the root s_0 corresponding to $m = 0$ is admissible, the other roots making $f'(s)$ equal to $2m\pi i$. Provided $|\arg w| < 4\pi$, this saddle point falls in the part of the domain of s where Stirling's formula is applicable. It is possible that there may be other saddle points near the negative real axis, but as the contour C_0 need not be drawn near these, this is immaterial. To evaluate $f(s)$ at this saddle point, substitute

$$s = s_0 \sim w^{\frac{1}{4}} + \tfrac{1}{2} - \tfrac{1}{4}\Sigma_j p_j$$

into equation (A. 5), and rearrange to obtain

$$f(s_0) \sim 2\log 2\pi + (s_0-\tfrac{1}{2})[\log \Pi_j(s_0+p_j-\tfrac{1}{2}) - 4] - s_0 \log w + \Sigma_j p_j[\log(s_0+p_j-\tfrac{1}{2}) - 1].$$

But
$$\Pi_j(s_0+p_j-\tfrac{1}{2}) = w$$

and
$$\log(s_0+p_j-\tfrac{1}{2}) \sim \log(w^{\frac{1}{4}}+p_j-\tfrac{1}{4}\Sigma_k p_k)$$
$$\sim \tfrac{1}{4}\log w + (p_j - \tfrac{1}{4}\Sigma_k p_k)/w^{\frac{1}{4}}.$$

Therefore

$$f(s_0) \sim 2\log 2\pi + (s_0-\tfrac{1}{2})(\log w - 4) - s_0 \log w$$
$$+ \Sigma_j p_j(\tfrac{1}{4}\log w - 1) + [\Sigma_j p_j^2 - \tfrac{1}{4}(\Sigma_j p_j)^2]/w^{\frac{1}{4}}.$$

If $|w|$ is sufficiently great, the last term is negligible. Therefore

$$f(s_0) \sim 2\log 2\pi - \tfrac{1}{2}\log w - 4(w^{\frac{1}{4}} - \tfrac{1}{4}\Sigma_j p_j) + \Sigma_j p_j(\tfrac{1}{4}\log w - 1) + O(w^{-\frac{1}{4}})$$
$$= 2\log 2\pi - (\tfrac{1}{2} - \tfrac{1}{4}\Sigma_j p_j)\log w - 4w^{\frac{1}{4}} + O(w^{-\frac{1}{4}}).$$

It follows that
$$\exp[f(s_0)] \sim (2\pi)^2 w^{-\frac{1}{2}-\frac{1}{4}\Sigma_j p_j} \exp(-4w^{\frac{1}{4}}) \exp[O(w^{-\frac{1}{4}})]. \tag{A. 8}$$

Also from (A. 7),
$$f''(s_0) \sim \Sigma_j 1/(s_0+p_j-\tfrac{1}{2}) \sim 4w^{-\frac{1}{4}} + O(w^{-\frac{1}{2}}). \tag{A. 9}$$

Hence applying the method of steepest descents,

$$I_0(w) \sim \frac{1}{2\pi i}\exp[f(s_0)]\int_{-i\infty}^{+i\infty} \exp[\tfrac{1}{2}f''(s_0)u^2]\,du$$
$$= \frac{1}{2\pi}\exp[f(s_0)]\int_{-\infty}^{+\infty} \exp[-\tfrac{1}{2}f''(s_0)v^2]\,dv$$
$$= \frac{1}{2\pi}\exp[f(s_0)]\sqrt{[2\pi/f''(s_0)]}.$$

Substituting the values of $\exp[f(s_0)], f''(s_0)$ from (A. 8) and (A. 9), we obtain the required integral

$$I_0(w) \sim \surd(2)\, \pi^{\frac{3}{2}} w^{-\frac{3}{8}+\frac{1}{4}\Sigma_j p_j} \exp[-4w^{\frac{1}{4}}].$$

This formula has been derived on the assumption that $|\arg w| < 4\pi$, but a more careful analysis would justify a limiting process to $\arg w = 4\pi$. Thus, if w is real and positive,

$$I_0(w) \sim \surd(2)\, \pi^{\frac{3}{2}} w^{-\frac{3}{8}+\frac{1}{4}\Sigma p_j} \exp[-4w^{\frac{1}{4}}] \exp[O(w^{-\frac{1}{4}})],$$

$$I_0(w\,e^{2i\pi}) \sim \surd(2)\, \pi^{\frac{3}{2}} \exp\left[\frac{i\pi}{2}(-\tfrac{3}{2}+\Sigma p_j)\right] w^{-\frac{3}{8}+\frac{1}{4}\Sigma p_j} \exp[-4iw^{\frac{1}{4}}] \exp[O(w^{-\frac{1}{4}})],$$

$$I_0(w\,e^{-2i\pi}) \sim \surd(2)\, \pi^{\frac{3}{2}} \exp\left[-\frac{i\pi}{2}(-\tfrac{3}{2}+\Sigma p_j)\right] w^{-\frac{3}{8}+\frac{1}{4}\Sigma p_j} \exp[4iw^{\frac{1}{4}}] \exp[O(w^{-\frac{1}{4}})],$$

$$I_0(w\,e^{4i\pi}) \sim \surd(2)\, \pi^{\frac{3}{2}} \exp[i\pi(-\tfrac{3}{2}+\Sigma p_j)]\, w^{-\frac{3}{8}+\frac{1}{4}\Sigma p_j} \exp[4w^{\frac{1}{4}}] \exp[O(w^{-\frac{1}{4}})].$$

Appendix 2. Table of arg $(iy)!$ and arg $(iy-\frac{1}{2})!$

The following table gives the arguments in degrees, being correct to the second decimal place.

y	arg $(iy)!$	arg $(iy-\frac{1}{2})!$
0·0	0	0
0·2	− 6·43	−21·32
0·4	−11·87	−37·34
0·6	−15·63	−47·35
0·8	−17·43	−52·73
1·0	−17·28	−54·72
1·2	−15·32	−54·13
1·4	−11·70	−51·46
1·6	− 6·62	−47·05
1·8	− 0·19	−41·16
2·0	7·43	−33·95
2·2	16·15	−25·57
2·4	25·87	−16·12
2·6	36·53	− 5·70
2·8	48·04	5·61
3·0	60·35	17·75
3·2	73·42	30·66
3·4	87·18	44·30
3·6	101·62	58·61
3·8	116·68	73·57
4·0	132·34	89·14

Appendix 3. Explanation of the relations (6·16)

Provided $g\pi$, $h\pi$ and $(h-g)\pi$ are $\gg 1$, equations (6·8) to (6·11) for the reflexion and transmission coefficients of region I show that, as $\epsilon \to 0$ (through positive values), waves incident from below are almost perfectly reflected, while waves incident from above are strongly absorbed.

The condition for the validity of the transformation formula for the hypergeometric function of argument v to one of argument $1/v$, used in § 6, is that

$$|\arg(-v)| < \pi.$$

$(-v)$ is thus interpreted as $v\exp(-i\pi)$, since

$$0 < \arg v = \epsilon < \tfrac{1}{2}\pi.$$

If $\epsilon < 0$, a different transformation formula is obtained; $-v$ must be interpreted as $v \exp(\mathrm{i}\pi)$. In fact, the hypergeometric function possesses branch-points at 1 and infinity (Copson 1935, chap. 10), and the transformation formula is valid in the whole plane only when cut along the real axis from 1 to $+\infty$. E_z thus has a singularity at $v = 1$, that is, where $X = 1 - \mathrm{i}Z$. If $Z = 0$, this occurs for a real value of the height z, and for values of z greater than this, the value of E_z is ambiguous (by this method of solution). In practice, Z is always greater than zero, so the singularity does not occur on the real z-axis and analytic continuation along the real z-axis is possible. Hence the formulae can only be regarded as valid if $\epsilon \to 0$ through positive values, the case $\epsilon = 0$ exactly being impossible physically, and, by this method of solution, ambiguous mathematically.

The following physical argument to explain the paradox was suggested by Dr K. G. Budden.

In (4·7) let
$$(1 + 1/l)\, E_z = F,$$
then
$$\mathrm{d}^2F/\mathrm{d}z^2 + k^2F/(1+l) = 0,$$
so the effective dielectric constant may be taken as
$$\mu^2 = \sin^2\theta + 1/(1+l),$$
or
$$\mu^2 - \sin^2\theta = \frac{\cos^2\theta - X/(1-\mathrm{i}Z)}{1 - X/(1-\mathrm{i}Z)},$$
substituting the value of l from (4·3).

If $Z \ll 1$, then μ^2 is almost real except when X is close to the values $\cos^2\theta$ or 1. As X increases from zero to $\cos^2\theta$, the real part of $\mu^2 - \sin^2\theta$ decreases from $\cos^2\theta$ to zero, and when $X = \cos^2\theta$, $\mu^2 - \sin^2\theta$ has the small imaginary value
$$\mu^2 - \sin^2\theta \doteqdot \mathrm{i}Z\cot^2\theta.$$

Thus waves incident from below may be regarded as reflected with little absorption from the height where $X = \cos^2\theta$, provided the change of X per unit wave-length is small, i.e. $\alpha \ll 1$.

If $\cos^2\theta < X < 1$, the real part of $\mu^2 - \sin^2\theta$ is negative; and when $X = 1$, $\mu^2 - \sin^2\theta$ has the large imaginary value
$$\mu^2 - \sin^2\theta \doteqdot -\frac{\mathrm{i}\sin^2\theta}{Z},$$
while for $X > 1$, the real part of $\mu^2 - \sin^2\theta$ is approximately $(X - \cos^2\theta)/(X-1)$, which is positive and greater than unity. Hence, as a wave incident from above descends, the effective dielectric constant increases, and the phase velocity decreases. At the height where $X = 1$, the dielectric constant is large and negative imaginary, implying great absorption. As Z diminishes, the thickness of the absorbing region becomes smaller, but the rate of absorption becomes more intense. As the formulae (6·8) to (6·11) show, $|r'|$ and $|t'|$ tend to finite limits as $Z \to 0$, and these limits are small if α is small.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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